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Oral examination:

Optimal scaling laws for domain patterns in thin ferromagnetic films with strong perpendicular anisotropy

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Abstract

The topic of this thesis are magnetic domains in thin ferromagnetic films with strong perpendicular anisotropy. Our starting point is Micromagnetics, a continuum model based on the principle of minimal energy. At its core is the micromagnetic energy functional, whose local minimizer represent the stable magnetization configurations of the ferromagnetic body.

Identifying a suitable thin film regime leads us to investigate a singular limit of the nonconvex and nonlocal micromagnetic energy functional. Our asymptotic analysis yields a scaling law for the typical domain size as a function of the film thickness and another material parameter. To prove an ansatz free lower bound of the energy, we extend an interpolation inequality first obtained in [26].

Moreover, we study a shape optimization problem that can be considered as a prototypical model for a single magnetic domain. We minimize the sum of the perimeter and the dipolar self-energy among subsets of \mathbb{R}^3 with prescribed volume. Upon proving that minimizers exist, we show that they are (\mathcal{L}^3 -equivalent to) connected open sets with smooth boundary. We furthermore establish a scaling law for the minimal energy in terms of the prescribed volume which yields further information about the shape of minimizers.

Zusammenfassung

Das Thema dieser Arbeit ist die Domänenstruktur in ferromagnetischen Filmen mit starker Anisotropie senkrecht zur Filmebene. Den Ausgangspunkt bildet das mikromagnetische Modell, welches auf einer Kontinuumsapproximation und dem Prinzip der minimalen Energie beruht. Zentraler Bestandteil des Modells ist das mikromagnetische Energiefunktional, dessen lokale Minimierer die stabilen Konfigurationen der Magnetisierung des ferromagnetischen Materials repräsentieren.

Die Identifikation eines geeigneten Regimes dünner Filme führt zur Untersuchung eines singulären Limes des nichtkonvexen und nichtlokalen mikromagnetischen Energiefunktional. Das asymptotische Verhalten der Energie impliziert ein Skalierungsgesetz für die typische Längenskala der erwarteten magnetischen Domänen als Funktion der Filmdicke und eines weiteren Materialparameters. Für den Beweis einer unteren Schranke der Energie wird eine Interpolationsungleichung aus [26] verschärft.

Des Weiteren wird ein Optimierungsproblem für die Form einer einzelnen magnetischen Domäne untersucht. Dabei wird die Summe des Oberflächeninhalts und der Demagnetisierungsenergie über geeignete Teilmengen des \mathbb{R}^3 mit vorgegebenem Volumen minimiert. Es wird bewiesen, dass Minimierer existieren und dass diese \mathcal{L}^3 -äquivalent zu offenen, zusammenhängenden Mengen mit glattem Rand sind. Der Beweis eines Skalierungsgesetzes für die minimale Energie als Funktion des vorgegebenen Volumens liefert weitere Informationen über die optimale Form von Minimierern.

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Introduction

This thesis is motivated by the phenomenon of spatially ordered magnetization patterns in thin ferromagnetic films with strong perpendicular anisotropy. These patterns usually consist of uniformly magnetized regions – *magnetic domains* – separated by transition regions called *domain walls* [44]. Experimentally observed patterns include so-called stripe, bubble or maze domain phases [88, 84, 43, 95, 77], depending on the geometry of the sample, external magnetic fields and other factors (see also Figure 1).

Magnetization patterns are not only of academic interest. Indeed, magnetic domains may be considered to “link the basic physical properties of a material with its macroscopic properties and applications” [44, p. vii]. Over the last decade, ferromagnetic films with perpendicular anisotropy and related multilayer constructions have played an indispensable role in data storage technologies [48]. Additionally, films and multilayer structures consisting of only a few atomic layers have received increased attention among experimentalists [49, 62, 94, 86, 87] due to possible applications in future spintronic devices [6].

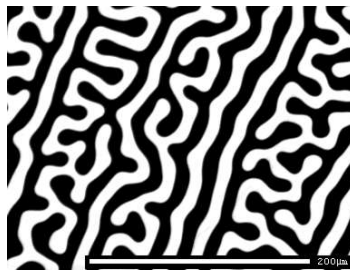


Figure 1: Faraday microscopy image depicting the domain structure of a garnet film with out-of-plane anisotropy (view from top; black/white indicates whether the magnetization points towards/away from the viewer). Reproduced from [69] with permission.

Beginning with the pioneering works of Landau and Lifshitz [64], Néel [79, 80] and Kittel [52], it turned out that magnetic domains may usually be modelled on the basis of energy considerations. From those works, the micromagnetic modeling framework emerged and has been reviewed by Kittel [53] and Brown [13]. Its core is the micromagnetic energy functional, whose local minimizers represent the stable magnetization configurations of the ferromagnetic body.

The micromagnetic energy has been the subject of intensive studies for various ferromagnetic systems in the physical and mathematical literature (see [44] and [28] for reviews). In particular, it was shown in [5] that strong uniaxial anisotropy of the material leads to magnetizations that are predominantly aligned with the so-called *easy axis* of the material. Moreover, for bulk samples with strong uniaxial anisotropy, the ground state has been studied in [18, 19, 81, 56]. Furthermore, numerous thin film regimes for the micromagnetic energy have been identified and investigated. However, the majority of those studies considers films where, unlike in our setting, the magnetization tends to lie in the film plane (see, e.g., [35, 15, 27, 74, 60, 26, 63, 47, 46, 45, 16]). Only a few studies in the mathematical literature consider films with magnetization perpendicular to the film plane. Their focus is on different parameter regimes [5, 38, 22], the effect of Dzyaloshinskii-Moriya interaction [76] or Skyrmions [70]. The ground state of ferromagnetic films with strong anisotropy perpendicular to the film plane has only been studied using ansatz based computations in the physical literature [61, 29, 51, 78]).

The first part of this thesis provides an ansatz free analysis of ferromagnetic films with strong anisotropy perpendicular to the film plane. We determine a scaling law for the minimal micromagnetic energy and investigate the configurations that achieve it. In particular, our analysis yields a scaling law for the length scale of domains (in an averaged sense) in terms of the film thickness and a material parameter.

When the ferromagnetic film is exposed to a critical external field, a phase transition between a complex, branching domain pattern and the uniform magnetization configuration occurs. In [56], Knüpfer and Muratov determined the scaling of the minimal energy for an external field close to saturation as well as the critical field strength. Moreover, they showed that a branching pattern

of thin and slender “needle-shaped” domains with magnetization opposing the applied field achieves the optimal scaling of the energy.

Based on the results of [56], we model a single one of the expected “needle-shaped” domains using a shape optimization problem in full space. The sum of the surface area and the dipolar self-energy is minimized among sets with prescribed volume. We show that local minimizers are (up to an \mathcal{L}^3 negligible set) connected open sets with smooth boundary. We furthermore establish a scaling law for the minimal energy in terms of the prescribed volume which yields further information about the shape of minimizers.

We believe that our analysis of this prototypical single domain model might also be of interest for other (highly anisotropic) pattern forming systems governed by the competition of interfacial and dipolar energies. As examples, we mention ferromagnetic gels [21] and certain dipolar Bose-Einstein condensates [85, 32], where “needle-shaped” configurations have been observed experimentally.

We note that our single domain model is the full space version of a Γ -limit of the micromagnetic energy obtained in [5]. Moreover, we want to mention the following related full space models, which are also the sum of an interfacial energy and a competing nonlocal energy term. The first example is a family of energies where the nonlocal term arises from the Riesz-type kernel $|z|^{\alpha-n}$ with $(\alpha \in (0, n))$. It has received a lot of interest recently [57, 58, 66, 20, 2, 10, 33, 50]. For $n = 3$ and $\alpha = 2$, one obtains the Gamow liquid drop model for atomic nuclei, where the nonlocal term can be understood as the Coulomb energy of a configuration with uniform charge density. See also [11] for a multi-phase version.

A second example is a model for elastic inclusions (in the framework of geometrically linearized elasticity). The scaling of its minimal energy was studied in [54]. See also [55] for a multi-phase version.

Before we describe the results of this thesis in more detail, we briefly introduce the micromagnetic model.

The micromagnetic model

Let the open, bounded set $\Omega \subset \mathbb{R}^3$ represent the region in space occupied by the ferromagnetic material. The quantity of interest is the magnetization, modeled as a unit vector field $m : \Omega \rightarrow \mathbb{S}^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$ that (locally) minimizes the micromagnetic energy functional. In a partially non-dimensionalized form, the micromagnetic energy is given by [64, 44]

$$\mathbb{E}(m) = \int_{\Omega} (l_{\text{ex}}^2 |\nabla m|^2 + Q (m_2^2 + m_3^2) + 2h_{\text{ext}} \cdot m) \, dx + \int_{\mathbb{R}^3} |h|^2 \, dx \quad (0.1)$$

for admissible configurations in the non-convex class

$$\mathbb{A} = \{m \in L^2(\mathbb{R}^3; \mathbb{R}^3) : |m(x)| = \chi_{\Omega} \text{ and } m|_{\Omega} \in H^1(\Omega; \mathbb{R}^3)\}. \quad (0.2)$$

In (0.2), and throughout this whole thesis, χ_{Ω} denotes the characteristic function of the set Ω . Let us briefly explain the terms in (0.1).

- (i) The first term is called the *exchange energy*. It is of quantum mechanical origin and describes the tendency of neighboring spins to be aligned [42]. The material parameter $l_{\text{ex}} > 0$ is called the *exchange length* and denotes a characteristic length scale of the material.
- (ii) The second term is called the *anisotropy energy*. It penalizes the deviation of the magnetization from the *easy axis* which we have taken to be e_1 throughout the whole thesis. The dimensionless material constant Q is also known as the *quality factor*. Our assumption of “strong” anisotropy is made precise by the inequality $Q > 1$ (see chapter 1 for a detailed explanation).
- (iii) The third term is called the *Zeeman energy* and incorporates the effects of an external magnetic field $h_{\text{ext}} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$.
- (iv) The last term is called the *stray field energy*. The *stray field* $h \in L^2(\mathbb{R}^3; \mathbb{R}^3)$ is determined by the static Maxwell’s equations in matter

$$\operatorname{div}(h + m) = 0 \quad \text{and} \quad \nabla \times h = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^3). \quad (0.3)$$

Hence, up to a sign, h coincides with the Helmholtz projection of m onto the space of gradients and the stray field energy amounts to the squared $\dot{H}^{-1}(\mathbb{R}^3)$ -norm of $\operatorname{div} m$. In particular, the energy depends on m in a nonlocal way.

Note that the constant configurations $m \equiv \pm e_1$ minimize the first two terms in the energy, but lead to contributions in $\operatorname{div} m$ on parts of the boundary of $\partial\Omega$ which are penalized by the stray field energy. A scaling argument indicates that, for sufficiently large samples, it should be advantageous to alternate between $m \approx e_1$ and $m \approx -e_1$, i.e. to form magnetic domains. The reader is referred to [44] for further details on the micromagnetic model and a survey of experimental techniques and results. A mathematical-minded introduction may be found in the survey [28].

Overview of the main results

We first explain the basic structure of this thesis which consists of three chapters. In turn, we will explain the main results of each chapter.

The first two chapters are concerned with properties of domain patterns in thin films with strong perpendicular anisotropy. The scaling of the minimal micromagnetic energy for such films is identified in chapter 1. It leads to a scaling law for the typical length scale of magnetic domains in such films. In chapter 2, we use this scaling law to initiate a finer analysis, corresponding to the next order in the Γ development of the energy. Upon heuristically simplifying the energy, we derive a nonlocal Γ -limit and study some of its properties. Whereas the magnetization is asymptotically two-dimensional in the first two chapters, Chapter 3 is concerned with three-dimensional magnetic domains that are expected in somewhat thicker films subject to a critical external field. We study a shape optimization problem (see (0.7)) for a single one of those domains.

We begin to explain the results of this thesis in more detail. The first chapter of this thesis is concerned with the asymptotic behavior of the micromagnetic energy for films of vanishing thickness and strong anisotropy perpendicular to the film plane (corresponding to $Q > 1$ in (0.1)). Starting from the full three-dimensional micromagnetic energy (0.1) and assuming periodicity in the film plane to avoid boundary effects, we show that the effective behavior is determined by the following two-dimensional functional $F_{\varepsilon,\lambda} : H^1(\mathbb{T}^2; \mathbb{S}^2) \rightarrow \mathbb{R}$,

given by

$$F_{\varepsilon,\lambda}[m] = \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2} |\nabla m|^2 + \frac{1}{2\varepsilon} (1 - m_1^2) \right) dx - \frac{\lambda}{4\pi |\log \varepsilon|} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \frac{|m_1(x) - m_1(y)|^2}{|x - y|^3} dy dx. \quad (0.4)$$

In (0.4), \mathbb{T}^2 denotes the square flat torus of unit side length, ε is the renormalized Bloch wall width and λ is the renormalized film thickness (see section 1.1 for the precise definitions). We remark that the double integral amounts to the squared homogeneous $H^{\frac{1}{2}}$ -norm. To simplify the exposition, we continue our discussion with the reduced energy $F_{\varepsilon,\lambda}$. However, we will prove analogous results for the full energy (0.1) by similar (but more involved) arguments.

The main part of our analysis is concerned with the asymptotic behavior of (0.4) as $\varepsilon \rightarrow 0$ for different values of $\lambda > 0$. Note that the last term in (0.4) occurs with a negative sign and hence prefers oscillations of m_1 . As it turns out, the value of the parameter λ is crucial. In fact, we will show that the asymptotic behavior changes at $\lambda = \lambda_c$, where $\lambda_c = \frac{\pi}{2}$, which is a singular point in the terminology of [12]. For $\lambda < \lambda_c$ the Γ -limit $F_{*,\lambda} := \Gamma(L^1)\text{-}\lim_{\varepsilon \rightarrow 0} F_{\varepsilon,\lambda}$ measures the length of the interface separating regions with $m \approx e_1$ and $m \approx -e_1$ and is given by (see also Theorem 1.2.5)

$$F_{*,\lambda}[m] = \begin{cases} \left(1 - \frac{\lambda}{\lambda_c}\right) \int_{\mathbb{T}^2} |\nabla m_1| dx, & \text{for } m \in BV(\mathbb{T}^2; \{\pm e_1\}), \\ +\infty, & \text{otherwise.} \end{cases}$$

Note that the last term in (0.4) leads to a reduction of the interfacial cost by $\frac{\lambda}{\lambda_c}$ compared to the classical result [5] for $\lambda = 0$. On the other hand, for $\lambda > \lambda_c$, the scaling of the minimal energy changes to (see also Theorem 1.2.6)

$$\min F_{\varepsilon,\lambda} \sim -\frac{\lambda \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}}}{|\log \varepsilon|} \xrightarrow{\varepsilon \rightarrow 0} -\infty. \quad (0.5)$$

Moreover, all sequences $(m_\varepsilon)_\varepsilon$ of configurations which achieve the optimal scaling $F_{\varepsilon,\lambda}[m_\varepsilon] \sim \min F_{\varepsilon,\lambda}$ become highly oscillatory in the sense that

$$\int_{\mathbb{T}^2} |\nabla (m_\varepsilon)_1| dx \sim \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}} \xrightarrow{\varepsilon \rightarrow 0} +\infty. \quad (0.6)$$

Estimate (0.6) may be interpreted as a scaling law for the typical distance of neighboring domain walls. The main difficulty in the proof is to find asymptotically optimal estimates for the $H^{\frac{1}{2}}$ -term. The lower bounds for $F_{\varepsilon,\lambda}$ rely

on an improved version of an interpolation inequality from [26]. Whereas our estimate is similar to the estimate in [26] when applied to monotone functions, it is significantly stronger for highly oscillatory functions such as configurations that minimize $F_{\varepsilon,\lambda}$ for $\lambda > \lambda_c$. In particular, our improvement is crucial to obtain (0.5) and (0.6).

Upon studying the asymptotic behavior of $F_{\varepsilon,\lambda}$, we carry out a similar program for the full micromagnetic energy (0.1). Here, additional difficulties arise in the approximation of the stray field energy and due to the transition from three-dimensional configurations to a two-dimensional limit.

Together with C. Muratov and H. Knüpfer a joint paper comprising the results of chapter 1 has been submitted.

In chapter 2, we use the scaling law obtained in chapter 1 to initiate a finer analysis, corresponding to the next order in the Γ -development of the micromagnetic energy. Our focus is on the case when the length of the film's unit-cell is much larger than, but still comparable to, the expected pattern size. Due to difficulties related in part to the diffuse interfaces, we are unable to carry out such a program for the full micromagnetic energy. Instead, we heuristically identify a related sharp interface model. During this process, we lose the strong regularizing effect of the exchange energy and it becomes crucial to exploit the natural regularization in the stray-field energy. Starting from the reduced sharp interface model, we prove the Γ -convergence towards a nonlocal functional.

Chapter 3 is motivated by questions on single magnetic domains in uniaxial ferromagnetic materials exposed to a critical external field and, more generally, the nucleation of domains in such samples. To this end, we consider a shape optimization problem for a single ferromagnetic domain $\Omega \subset \mathbb{R}^3$, represented by its characteristic function $\chi_\Omega \in BV(\mathbb{R}^3; \{0, 1\})$. We assume that χ_Ω is a (local) minimizer of the energy

$$\mathcal{E}(\chi) = \int_{\mathbb{R}^3} |\nabla \chi| \, dx + \int_{\mathbb{R}^3} |h_\chi|^2 \, dx \quad (0.7)$$

among configurations with prescribed volume $\int_{\mathbb{R}^3} \chi \, dx = V$. In the last term

in (0.7), $h_\chi \in L^2(\mathbb{R}^3; \mathbb{R}^3)$ represents the stray field, determined by (cf. (0.3))

$$h_\chi := -\nabla\Phi, \quad \text{where } \Phi \in \dot{H}^1(\mathbb{R}^3) \text{ solves } \operatorname{div}(-\nabla\Phi + \chi e_1) = 0.$$

We first confirm that minimizers of (0.7) exist for all volumes V . The proof uses the concentration compactness principle and the sublinear scaling of the minimal energy. Next, we turn to the regularity of local minimizers and prove that they are (up to \mathcal{L}^3 -equivalence) bounded open sets with smooth boundary. The proof is based on the $C^{1,\alpha}$ -regularity results for quasi-minimizers of the perimeter functional. Additionally, we exploit stationarity of the energy with respect to inner variations. Furthermore, classical results from potential theory imply that the corresponding stray field h_χ is in $L^\infty(\mathbb{R}^3; \mathbb{R}^3)$. We then turn to topological properties of local minimizers. In particular, we prove that certain regular representatives of local minimizers of (0.7) are connected. Finally, we identify the scaling of the minimal energy in terms of the prescribed volume V which turns out to be

$$\min_{\substack{\chi \in BV(\mathbb{R}^3; \{0,1\}), \\ \int \chi \, dx = V}} \mathcal{E}(\chi) \sim \begin{cases} V^{\frac{2}{3}} & \text{for } V \leq 1, \\ V^{\frac{5}{7}} (\log eV)^{\frac{1}{7}} & \text{for } V > 1. \end{cases}$$

The upper bound for large V is obtained by (the characteristic function of) prolate spheroids with length $V^{\frac{3}{7}} (\log V)^{\frac{2}{7}}$ along the e_1 -direction and radius $V^{\frac{2}{7}} (\log V)^{-\frac{1}{7}}$ in the plane perpendicular to e_1 . The proof of the lower bound is based on ideas and a geometric construction from [17].

We take a moment to compare the settings in chapters 1 to 3. Their unifying theme is that an energy – essentially given by the sum of interfacial and dipolar self-energy – is minimized among highly anisotropic magnetization configurations which are (approximately) aligned with the e_1 -axis. In all settings, the anisotropy ultimately originates from the leading order of (0.1). But this mechanism is replaced by a constraint in chapters 2 and 3. This is of course related to the passage from a diffuse to a sharp interface description, which also removes an additional length scale given by the width of interfaces. The main difference between the settings in chapters 1 and 2 versus chapter 3 is the geometric constraint of the film in chapters 1 and 2 and the critical external field which leads to the volume constraint in chapter 3.

Notation: For $x \in \mathbb{R}^n$, $n \geq 2$ we write $x = (x_1, x')$, where x' is the projection of x onto the last $(n - 1)$ components. For $\mu > 0$ and $\Omega \subset \mathbb{R}^n$, we write $\mu\Omega = \{\mu x : x \in \Omega\}$ to denote the isotropic rescaling of Ω by μ . We write χ_Ω to denote the characteristic function of Ω . The open Euclidean ball with center x_0 and radius r is denoted by $B_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$ and we set $B_r := B_r(0)$. The symbol $\delta_{i,k}$ denotes the Kronecker Delta with $\delta_{i,k} = 1$ if $i = k$ and $\delta_{i,k} = 0$ otherwise.

The n -dimensional Lebesgue measure of a measurable set $\Omega \subset \mathbb{R}^n$ is denoted by $|\Omega|$ and if B_1 is the unit ball in \mathbb{R}^n we set $\omega_n = |B_1|$. The k -dimensional Hausdorff measure is denoted by \mathcal{H}^k .

Our notation for function spaces follows [30, 31] to which we also refer as references. This includes the Hölder spaces $C^{k,\alpha}$ of functions with α -Hölder continuous derivatives up to order k , the space of functions with bounded variation BV , and the Sobolev spaces $W^{k,p}$ of functions with weak partial derivatives up to order k in L^p . For the latter, we set $H^k := W^{k,2}$ when $p = 2$. Additionally, if $\Omega \subset \mathbb{R}^n$ is open and $Y \subset \mathbb{R}^m$, we write $f \in W^{k,p}(\Omega; Y)$ to denote that $f \in W^{k,p}(\Omega, \mathbb{R}^m)$ and $f(x) \in Y$ for almost every $x \in \Omega$ (and likewise for $BV(\Omega, Y)$).

Let $k \geq 0$ be an integer, $\alpha \in (0, 1]$ and let $\Omega \subset \mathbb{R}^n$ be open and bounded. We say that $\partial\Omega$ is a $C^{k,\alpha}$ boundary if, for each point $y \in \partial\Omega$, there exists $r > 0$ and a function $\gamma \in C^{k,\alpha}(\mathbb{R}^{n-1})$ such that – upon rotating and relabeling the coordinate axes if necessary – we have

$$\Omega \cap B_r(y) = \{(x_1, x') \in B_r(y) : x_1 > \gamma(x')\}. \quad (0.8)$$

Furthermore, $\partial\Omega$ is smooth if $\gamma \in C^\infty$. If $\partial\Omega$ is a $C^{k,\alpha}$ boundary, we define the Hölder space $C^{k,\alpha}(\partial\Omega)$ in terms of local coordinates and refer to [73, Chapter I.1] for further details.

The expression $f(x) \lesssim g(x)$ means that there exists a universal constant $C > 0$ such that the inequality $f(x) \leq Cg(x)$ holds for every x . The symbol \gtrsim is defined analogously with \geq instead of \leq and we write \sim if both \lesssim and \gtrsim hold. The previous relations are to be distinguished from \approx which is only used in heuristic arguments and denotes “approximately equal” (in an unspecified

sense).

The flat torus with side length $\ell > 0$ is denoted by $\mathbb{T}_\ell^n := (\mathbb{R}^n / \ell \mathbb{Z}^n)$ and we abbreviate $\mathbb{T}^n := \mathbb{T}_1^n$. We frequently identify functions $u : \mathbb{T}_\ell^n \rightarrow \mathbb{R}$ with ℓ -periodic functions $v : \mathbb{R}^n \rightarrow \mathbb{R}$ by means of the natural projection $\pi : \mathbb{R}^n \rightarrow \mathbb{T}_\ell^n$, i.e. when $u = v \circ \pi$ holds we identify u with v .

For $u \in L^1((0, t) \times \mathbb{T}_\ell^2)$ we write $\bar{u} \in L^1(\mathbb{T}_\ell^2)$ to denote the e_1 -average, given by

$$\bar{u}(x') = \frac{1}{t} \int_0^t u(x_1, x') \, dx_1. \quad (0.9)$$

Moreover, for every $v \in L^1(\mathbb{T}_\ell^2)$ we write $\chi_{(0,t)} v \in L^1((0, t) \times \mathbb{T}_\ell^2)$ to denote the function $(\chi_{(0,t)} v)(x_1, x') = \chi_{(0,t)}(x_1) v(x')$ for $(x_1, x') \in (0, t) \times \mathbb{T}_\ell^2$.

For future reference, we now fix the constants in the definition of the Fourier coefficients. For $f \in L^2(\mathbb{T}_\ell^2)$, we write

$$\widehat{f}_k = \int_{\mathbb{T}_\ell^2} e^{-ik \cdot x} f(x) \, dx, \quad \text{where } k \in \frac{2\pi}{\ell} \mathbb{Z}^2. \quad (0.10)$$

The inverse Fourier transform is then given by

$$f(x) = \frac{1}{\ell^2} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} e^{ik \cdot x} \widehat{f}_k, \quad (0.11)$$

where convergence is understood in the $L^2(\mathbb{T}_\ell^2)$ sense. Parseval's Theorem then states that

$$\int_{\mathbb{T}_\ell^2} f^*(x) g(x) \, dx = \frac{1}{\ell^2} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} \widehat{f}_k^* \widehat{g}_k \quad \text{for } f, g \in L^2(\mathbb{T}_\ell^2), \quad (0.12)$$

where “*” denotes complex conjugation. Furthermore, we use the symbol $\nabla^s u$ to denote

$$\int_{\mathbb{T}_\ell^2} |\nabla^s u|^2 \, dx := \frac{1}{\ell^2} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} |k|^{2s} |\widehat{u}_k|^2 \quad (0.13)$$

for $s \in \mathbb{R}$. For $s = \frac{1}{2}$ we will also use the following well-known real space representation of the (square of the) homogeneous $H^{\frac{1}{2}}(\mathbb{T}_\ell^2)$ -norm

$$\int_{\mathbb{T}_\ell^2} |\nabla^{\frac{1}{2}} u|^2 \, dx = \frac{1}{4\pi} \int_{\mathbb{T}_\ell^2} \int_{\mathbb{R}^2} \frac{|u(x+y) - u(x)|^2}{|y|^3} \, dy \, dx. \quad (0.14)$$

For the convenience of the reader, a proof of the equivalence is provided in Lemma A.4 in Appendix A.

Chapter 1

Domains in ultrathin films

In this chapter, we are interested in deriving a reduced two-dimensional model for ultrathin ferromagnetic films with strong perpendicular anisotropy. Moreover, we characterize low energy states in films of large spatial extent. Our starting point is the three-dimensional micromagnetic energy functional (0.1). Since our focus is on materials with strong perpendicular anisotropy, we assume that the parameter Q in (0.1) is greater than 1 (the value 1 is explained below). The high anisotropy leads to magnetizations that are predominantly perpendicular to the film plane. It is well-known that such materials feature magnetizations that consist of one or many regions of nearly constant magnetization, called magnetic domains, separated by interfaces, called domain walls. We identify the critical scaling for the size of the sample where a transition from single domain states to multidomain states occurs. Moreover, we analyze the asymptotic behavior of the energy in the two regimes separated by this transition. In the subcritical regime, the global minimizers are the single domain states $m = \pm e_1$. We derive the asymptotic behavior of the energy in this regime in the framework of Γ -convergence. The reduced energy turns out to be much simpler than the full energy, in particular, it is two-dimensional and local. In the supercritical regime, which lies beyond the transition towards multidomain configurations, we establish the scaling of the energy (up to a multiplicative constant) and characterize sequences that achieve this scaling. Our analysis shows that the magnetization in this regime consists of several

domains and suggests that the typical distance between domain walls scales as

$$\text{typical domain size } S \sim \frac{e^{\frac{2\pi l_{\text{ex}}\sqrt{Q-1}}{T}}}{\sqrt{Q-1}} l_{\text{ex}} \quad (1.1)$$

where T is the thickness of the film.

Although additional physical effects become important in ultrathin films (e.g. Dzyaloshinskii-Moriya interaction [9, 83]), we believe that our results are sufficiently robust and carry over at least qualitatively to more general models that incorporate these effects (see, e.g., [76]).

A reduction of the full three-dimensional micromagnetic energy to a local two-dimensional model in the thin film limit was first established rigorously in [38]. Subsequently, several thin film regimes for magnetically soft materials have been identified and analyzed, see, e.g., [15, 27, 74, 60, 63, 46]. However, since we consider materials with high perpendicular anisotropy, our setting is considerably different, as we now explain. For thin films of the form $\Omega = (0, t) \times \mathbb{T}^2$, the leading order contribution of the stray field energy penalizes the out-of-plane component of the magnetization. Neglecting boundary effects, we have (see, e.g., Theorem 1.5.2)

$$\left| \int_{\mathbb{R} \times \mathbb{T}^2} |h|^2 \, dx - \int_{(0,t) \times \mathbb{T}^2} m_1^2 \, dx \right| \lesssim t \int_{(0,t) \times \mathbb{T}^2} |\nabla m|^2 \, dx.$$

To our knowledge, the first result in this direction is contained in [38]. In the absence of high perpendicular anisotropy or a sufficiently strong external field (as in the previously mentioned papers) the micromagnetic energy forces the out-of-plane component m_1 to vanish asymptotically. In our setting, the anisotropy energy $Q \int_{\Omega} (m_2^2 + m_3^2) \, dx = Q \int_{\Omega} (1 - m_1^2) \, dx$ is however sufficiently strong (recall that $Q > 1$) such that low energy configurations require $m \approx \pm e_1$ on most of the domain. Hence, domain patterns may asymptotically be described by the scalar quantity m_1 . We note that the reduction towards a two-dimensional model of a closely related scalar problem involving a Ginzburg-Landau energy with dipolar interactions has recently been analyzed in [75].

The behavior of the material changes when the film can no longer be considered to be thin. In [18, 19] the scaling of the minimal energy was identified (for a sharp interface version of the micromagnetic model). The estimates have been refined to an asymptotic equality in [81]. Magnetizations with optimal energy involve so-called branching domain patterns which become finer and finer as they approach the boundary of the sample. When the ferromagnetic sample is exposed to a critical external field, a transition between a uniform and a branching domain pattern occurs. The critical field strength and the scaling of the micromagnetic energy for this regime were derived in [56]. In our regime, the thickness of the film is so small that this does not only exclude the branching patterns that occur in bulk samples, but actually forces the magnetization to become constant in the direction normal to the film plane.

1.1 Setting

In order to non-dimensionalize the micromagnetic energy, we express lengths as multiples of the exchange length l_{ex} and rescale (effectively this amounts to setting $l_{\text{ex}} = 1$). We are interested in thin ferromagnetic films of uniform (non-dimensionalized) thickness t . For simplicity, we assume that the film extends infinitely in the film plane and that its magnetization is periodic in both in-plane coordinates with period ℓ . This means that we neglect boundary effects in the case of a finite sample of large spatial extent.

The film is composed of a uniaxial ferromagnetic material whose easy axis is perpendicular to the film plane, i.e. parallel to e_1 . Furthermore, we assume that the external field h_{ext} is parallel to e_1 and hence independent of x_1 (due to $\nabla \cdot h_{\text{ext}} = 0$). By a slight abuse of notation, from now on, we consider $h_{\text{ext}} : \mathbb{T}_\ell^2 \rightarrow \mathbb{R}$ as a scalar function. The non-dimensionalized energy per unit-cell $(0, t) \times \mathbb{T}_\ell^2$ then reads

$$E(m) := \int_{(0,t) \times \mathbb{T}_\ell^2} (|\nabla m|^2 + Q(m_2^2 + m_3^2) - 2m_1 h_{\text{ext}}) \, dx + \int_{\mathbb{R} \times \mathbb{T}_\ell^2} |h|^2 \, dx. \quad (1.2)$$

In the last term of (1.2), the stray field is the unique distributional solution $h \in L^2(\mathbb{R} \times \mathbb{T}_\ell^2; \mathbb{R}^3)$ of

$$\nabla \times h = 0 \quad \text{and} \quad \nabla \cdot (h + m) = 0 \quad \text{in } \mathbb{R} \times \mathbb{T}_\ell^2, \quad (1.3)$$

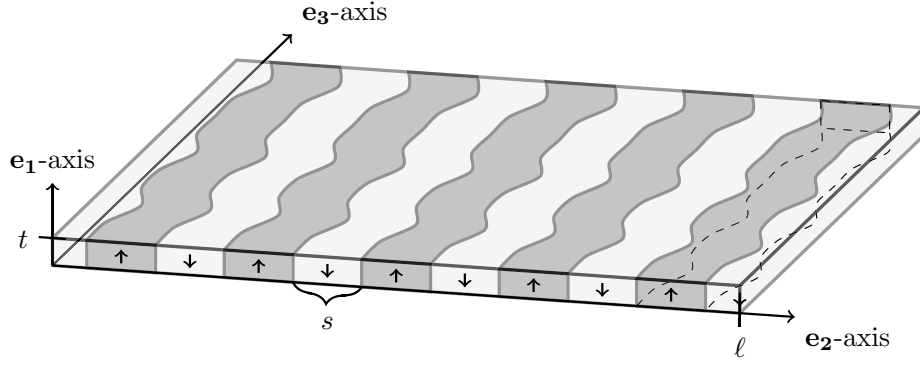


Figure 1.1: Typical magnetization pattern ("stripe pattern") in a unit cell $(0, t) \times \mathbb{T}_\ell^2$ of the ferromagnetic film. The arrows represent the value of the magnetization $m(x)$ at x , which is approximately constant across regions of the same color. The domains are separated by continuous domain walls of vanishing thickness, depicted as lines.

where $m \in H^1((0, t) \times \mathbb{T}_\ell^2)$ is extended by zero to $\mathbb{R} \times \mathbb{T}_\ell^2$. Hence, up to a sign, h equals the Helmholtz projection of m onto the space of gradients. We also use the notation $h = h[m]$ to denote the solution of (1.3).

Note that (1.2) depends on the three dimensionless parameters ℓ, t and Q . We are interested in the asymptotic behavior of the energy in (1.2) for thin films (i.e. $t \ll 1$) with large extension in the film plane (i.e. $\ell \gg 1$) and high anisotropy (i.e. $Q > 1$).

1.1.1 Identification of the regimes and the reduced energy F

In this section, we motivate the rigorous results contained in section 1.2. We use heuristic arguments to identify the scaling of the transition between monodomain and multi-domain states, and to explain how the micromagnetic energy E in (1.2) is related to the two-dimensional reduced energy F in (0.4). Roughly speaking, we will argue that (upon rescaling) F is a prototype for the next-to-leading-order term in the Γ -development of E , cf. [4].

To simplify the exposition, we neglect the energy contribution due to the external field h_{ext} . Furthermore we make two assumptions (for this section only), stated below. These assumptions are actually consequences of the thin film regime (see (1.89) and Theorem 1.5.2). Our assumptions are:

(i) The magnetization m is constant in the direction normal to the film, i.e.

$$m(x_1, x') = \chi_{(0,t)}(x_1) \overline{m}(x') \quad \text{for } x = (x_1, x') \in (0, t) \times \mathbb{T}_\ell^2. \quad (\text{i})$$

(ii) The stray field energy can be approximated by

$$\int_{\mathbb{R} \times \mathbb{T}_\ell^2} |h[m]|^2 \, dx \approx t \int_{\mathbb{T}_\ell^2} \overline{m}_1^2 \, dx - \frac{t^2}{2} \int_{\mathbb{T}_\ell^2} |\nabla^{\frac{1}{2}} \overline{m}_1|^2 \, dx. \quad (\text{ii})$$

Assumption (i) can be understood as a consequence of the vanishing thickness of the film which is smaller than the thickness of optimal domain walls (so-called Bloch walls).

We will now motivate Assumption (ii). For magnetizations that are constant in the normal direction of the film, i.e. $m(x_1, x') = \chi_{(0,t)}(x_1) \overline{m}(x')$, it is well-known that the stray field energy splits into a contribution due to the normal component \overline{m}_1 and a contribution due to the in-plane divergence $\nabla' \cdot \overline{m}' = \partial_2 \overline{m}_2 + \partial_3 \overline{m}_3$, see, e.g., [3, 35]. With the aid of the Fourier transform, a direct calculation yields (see also Theorem 1.5.2)

$$\begin{aligned} \int_{\mathbb{R} \times \mathbb{T}_\ell^2} |h[m]|^2 \, dx &= \frac{1}{\ell^2} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} t \sigma(t|k|) |\widehat{\overline{m}}_{1,k}|^2 \\ &\quad + \frac{1}{\ell^2} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} t (1 - \sigma(t|k|)) \left| \frac{k}{|k|} \cdot \widehat{\overline{m}}'_k \right|^2, \end{aligned} \quad (1.4)$$

where the Fourier multiplier σ is given by $\sigma(s) = \frac{1-e^{-s}}{s}$. In the electrostatics analogy, the first term on the right hand side can be understood as the contribution of surface charges proportional to \overline{m}_1 at the top and bottom surface of the film, whereas the second term describes the contribution due to volume charges proportional to $\nabla' \cdot \overline{m}'$. Since the strong anisotropy requires $|m_1| \approx 1$ on most of the domain, a scaling argument indicates that only the contribution due to m_1 is relevant. Indeed, since $|1 - \sigma(t|k|)| \leq t|k| \leq t(1 + |k|^2)$ the contribution due to m' may be estimated by the exchange and anisotropy energy at lower order

$$\frac{1}{\ell^2} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} t (1 - \sigma(t|k|)) \left| \frac{k}{|k|} \cdot \widehat{\overline{m}}'_k \right|^2 \leq t^2 \int_{\mathbb{T}_\ell^2} (|\nabla m|^2 + |m'|^2) \, dx. \quad (1.5)$$

The right hand side of (ii) is obtained by neglecting the second term on the right hand side of (1.4) and approximating $\sigma(s) \approx 1 - \frac{s}{2}$ in the first term (see Theorem 1.5.2 for a rigorous version).

With (i), (ii) and $h_{\text{ext}} = 0$, the energy (1.2) can now be written as

$$\begin{aligned} E(m) \approx & t \int_{\mathbb{T}_\ell^2} (|\nabla \bar{m}|^2 + Q (\bar{m}_2^2 + \bar{m}_3^2)) \, dx \\ & + t \int_{\mathbb{T}_\ell^2} \bar{m}_1^2 \, dx - \frac{t^2}{2} \int_{\mathbb{T}_\ell^2} |\nabla^{\frac{1}{2}} \bar{m}_1|^2 \, dx. \end{aligned} \quad (1.6)$$

We use the constraint $|\bar{m}| = 1$ to combine the leading order stray-field energy term with the anisotropy energy

$$\int_{\mathbb{T}_\ell^2} \bar{m}_1^2 \, dx + \int_{\mathbb{T}_\ell^2} Q (\bar{m}_2^2 + \bar{m}_3^2) \, dx = \ell^2 + \int_{\mathbb{T}_\ell^2} (Q - 1) (\bar{m}_2^2 + \bar{m}_3^2) \, dx. \quad (1.7)$$

Inserting (1.7) into (1.6) allows to extract the leading order constant

$$\begin{aligned} E(m) \approx & \ell^2 t + t \int_{\mathbb{T}_\ell^2} (|\nabla \bar{m}|^2 + (Q - 1) (\bar{m}_2^2 + \bar{m}_3^2)) \, dx \\ & - \frac{t^2}{2} \int_{\mathbb{T}_\ell^2} |\nabla^{\frac{1}{2}} \bar{m}_1|^2 \, dx. \end{aligned}$$

Upon rescaling \mathbb{T}_ℓ^2 to the fixed domain \mathbb{T}^2 and renormalizing the energy, we obtain

$$\begin{aligned} \frac{E(m(\ell \cdot)) - \ell^2 t}{\ell t \sqrt{Q - 1}} \approx & \int_{\mathbb{T}^2} \left(\frac{1}{\ell \sqrt{Q - 1}} |\nabla \bar{m}|^2 + \ell \sqrt{Q - 1} (\bar{m}_2^2 + \bar{m}_3^2) \right) \, dx \\ & - \frac{t}{2 \sqrt{Q - 1}} \int_{\mathbb{T}^2} |\nabla^{\frac{1}{2}} \bar{m}_1|^2 \, dx. \end{aligned} \quad (1.8)$$

In order to determine the critical scaling where minimizers of (1.8) cease to be constant and start to oscillate, we ask for which ℓ, t and Q it is possible to control the last term by the first integral

$$\begin{aligned} & \frac{t}{2 \sqrt{Q - 1}} \int_{\mathbb{T}^2} |\nabla^{\frac{1}{2}} \bar{m}_1|^2 \, dx \\ & \stackrel{?}{\lesssim} \int_{\mathbb{T}^2} \left(\frac{1}{\ell \sqrt{Q - 1}} |\nabla \bar{m}|^2 + \ell \sqrt{Q - 1} (\bar{m}_2^2 + \bar{m}_3^2) \right) \, dx. \end{aligned}$$

We make a one-dimensional ansatz \tilde{m} corresponding to N domains separated by smooth domain walls of width ε , see Figure 1.2. For the nonlocal term, a

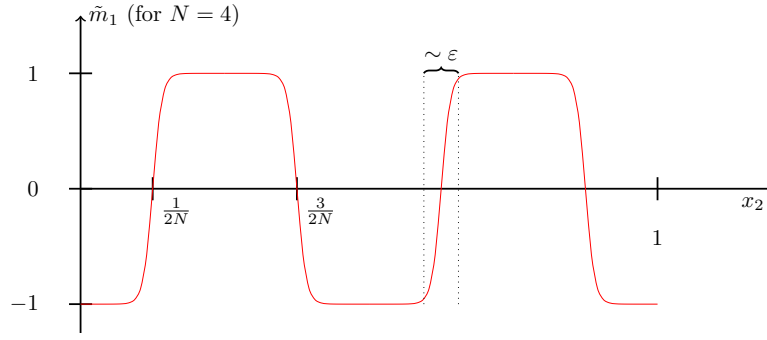


Figure 1.2: One-dimensional ansatz modeling a stripe pattern.

straightforward computation yields (see Lemma 1.4.2)

$$\begin{aligned} \int_{\mathbb{T}^2} |\nabla^{\frac{1}{2}} \tilde{m}_1|^2 \, dx &= \frac{1}{4\pi} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \frac{|\tilde{m}_1(x+z) - \tilde{m}_1(x)|^2}{|z|^3} \, dz \, dx \\ &\approx \frac{4}{\pi} \log \left(\frac{1}{\varepsilon N} \right) N. \end{aligned}$$

Since the nonlocal term depends only logarithmically on the transition layer, we optimize the width and internal structure of the transition layer for the first two terms in the energy by choosing $\varepsilon = \frac{1}{\ell\sqrt{Q-1}}$. For the corresponding Bloch wall profiles [44], we obtain

$$\begin{aligned} \int_{\mathbb{T}^2} \left(\frac{1}{\ell\sqrt{Q-1}} |\nabla \tilde{m}|^2 + \ell\sqrt{Q-1} (\tilde{m}_2^2 + \tilde{m}_3^2) \right) \, dx \\ \approx 2 \int_{\mathbb{T}^2} |\nabla \tilde{m}_1| \, dx \approx 4N. \end{aligned}$$

Hence

$$\frac{E(\tilde{m}(\ell \cdot)) - \ell^2 t}{\ell t \sqrt{Q-1}} \approx N \left(4 - \frac{2t}{\pi \sqrt{Q-1}} \log \left(\frac{\ell \sqrt{Q-1}}{N} \right) \right). \quad (1.9)$$

The (renormalized) energy of our ansatz (1.9) becomes negative, i.e. smaller than the energy of the constant configurations $m \equiv \pm e_1$, if $8\sqrt{Q-1} < \frac{4}{\pi} t \log \left(\frac{\ell \sqrt{Q-1}}{N} \right)$. By monotonicity in N , we expect that the critical scaling occurs for $N = 1$ and $t \sim t_c$, where

$$t_c \approx \frac{2\pi\sqrt{Q-1}}{\log(\ell\sqrt{Q-1})} \quad (1.10)$$

is the critical thickness of the onset of multidomain states.

Inserting (1.10) into (1.8) and abbreviating

$$\varepsilon = \frac{1}{\ell\sqrt{Q-1}}, \quad \lambda = \frac{t \log(\ell\sqrt{Q-1})}{4\sqrt{Q-1}},$$

we are led to study the asymptotic behavior for $\varepsilon \rightarrow 0$ of the family of functionals $F_{\varepsilon,\lambda} : L^1(\mathbb{T}^2; \mathbb{S}^2) \rightarrow \mathbb{R} \cup \{+\infty\}$, given by

$$F_{\varepsilon,\lambda}(m) = \begin{cases} \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2} |\nabla m|^2 + \frac{1}{2\varepsilon} (1 - m_1^2) \right) dx & m \in H^1(\mathbb{T}^2; \mathbb{S}^2), \\ -\frac{\lambda}{|\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla^{\frac{1}{2}} m_1|^2 dx, & \\ +\infty & \text{otherwise,} \end{cases} \quad (1.11)$$

where $\lambda \sim 1$ is a fixed parameter and with $\min E \approx \ell^2 t + 2\ell t \sqrt{Q-1} \min F_{\varepsilon,\lambda}$.

Remark 1.1.1. (Natural cut-off in the stray field energy) For thin films, the exponential decay of the Fourier multiplier in (1.4) leads to a natural regularization of the stray field energy. Instead of (ii), we could have used the alternative approximation

$$\begin{aligned} & \int_{\mathbb{R} \times \mathbb{T}_\ell^2} |h[m]|^2 dx \\ & \approx t \int_{\mathbb{T}_\ell^2 \times (0,t)} \bar{m}_1^2 dx - \frac{t^2}{8\pi} \int_{\mathbb{T}_\ell^2} \int_{\mathbb{R}^2 \setminus B_t} \frac{|\bar{m}_1(x+z) - \bar{m}_1(x)|^2}{|z|^3} dz dx, \end{aligned}$$

where the region $|z| < t$ is excluded in the last integral. However, our approximations in (ii) and in Theorem 1.5.2 ignore this cut-off. We will now explain that due to periodicity, this cut-off is not relevant in our setting. Roughly speaking, the reason is that the length scale of the cut-off is much smaller than the width of domain walls, which is the smallest length scale on which \bar{m} varies. More precisely, we have (see Lemma 1.3.1)

$$\begin{aligned} t^2 \int_{\mathbb{T}_\ell^2} \int_{B_t} \frac{|\bar{m}_1(x+z) - \bar{m}_1(x)|^2}{|z|^3} dx dz & \lesssim t^3 \int_{\mathbb{T}_\ell^2} |\nabla \bar{m}_1|^2 dx \\ & \lesssim t^2 \int_{(0,t) \times \mathbb{T}_\ell^2} |\nabla m|^2 dx, \end{aligned} \quad (1.12)$$

so that the effect due to the cut-off is controlled by the exchange energy at lower order. Here we have implicitly used that the film is periodic and hence does not have boundaries. On the other hand, if the ferromagnetic material is modeled by a finite domain $(0, t) \times \Omega$, exploiting the cut-off in the stray field energy becomes crucial: At the boundary $\partial\Omega$, the out-of-plane component \bar{m}_1 should jump so that $\|\bar{m}_1\|_{H^{\frac{1}{2}}(\mathbb{R}^2)}$ would be infinite. Since the exchange energy is oblivious to this jump at the boundary, (1.12) does not hold for Ω instead of \mathbb{T}_ℓ^2 .

1.2 Main results

Our main result is the identification of two thin-film regimes separated by a transition and the derivation of the asymptotic behavior of the energy in the regimes. We will state the results for the full energy E in Section 1.2.1 and for the reduced energy F in Section 1.2.2.

1.2.1 Results for the full energy E

In terms of ℓ , t and Q , the regimes may be expressed by

$$Q > 1, \quad \ell \gg 1 \quad \text{and} \quad \frac{t |\log(\ell \sqrt{Q-1})|}{4\sqrt{Q-1}} = \lambda$$

and $\lambda_c := \pi/2$, where

- $\lambda < \lambda_c$ corresponds to the subcritical regime featuring single domain states,
- $\lambda = \lambda_c$ corresponds to the transition,
- $\lambda_c < \lambda < \gamma \frac{|\log(\ell \sqrt{Q-1})|}{Q-1}$, for some universal $\gamma > 0$, corresponds to the multidomain state.

The upper bound $\lambda < \gamma \frac{|\log(\ell \sqrt{Q-1})|}{Q-1}$ is necessary because we do not know whether magnetizations are approximately two-dimensional beyond this threshold.

It is convenient to rescale the domain of the ferromagnetic film to a fixed domain by means of the anisotropic transformation

$$(0, t) \times \mathbb{T}_\ell^2 \rightarrow (0, 1) \times \mathbb{T}^2 \quad \text{with} \quad (x_1, x_2, x_3) \mapsto \left(\frac{x_1}{t}, \frac{x_2}{\ell}, \frac{x_3}{\ell} \right),$$

and study the renormalized energy $J : L^1((0, 1) \times \mathbb{T}^2; \mathbb{S}^2) \rightarrow \mathbb{R} \cup \{+\infty\}$, defined by

$$J(m) := \begin{cases} \frac{E(m(t \cdot, \ell \cdot, \ell \cdot)) - \ell^2 t}{\ell t \sqrt{Q-1}} & \text{for } m \in H^1((0, 1) \times \mathbb{T}^2 \times; \mathbb{S}^2), \\ +\infty & \text{otherwise.} \end{cases} \quad (1.13)$$

The asymptotic behavior of (1.13) in the subcritical regimes is characterized in the following theorem.

Theorem 1.2.1 (Subcritical regime). *Let $\lambda_c := \frac{\pi}{2}$, $\lambda \in [0, \lambda_c)$, $Q > 1$ and $(\ell_k, t_k, h_{\text{ext},k})_{k \in \mathbb{N}}$ be a sequence with*

$$\ell_k \rightarrow \infty, \quad \frac{t_k |\log(\ell_k \sqrt{Q-1})|}{4\sqrt{Q-1}} = \lambda \quad \text{and} \quad \frac{\ell_k}{\sqrt{Q-1}} h_{\text{ext},k}(\ell_k \cdot) \rightarrow g \quad (1.14)$$

for some $g \in L^1(\mathbb{T}^2)$ and for all $k \in \mathbb{N}$. Then the sequence of renormalized energies $(J_k)_{k \in \mathbb{N}}$, defined by (1.13) with $(\ell, t, h_{\text{ext}})$ replaced by $(\ell_k, t_k, h_{\text{ext},k})$, satisfies

(i) **Compactness:** *For every sequence $(m_k)_{k \in \mathbb{N}}$ in $L^1((0,1) \times \mathbb{T}^2; \mathbb{S}^2)$ with*

$$\limsup_{k \rightarrow \infty} J_k(m_k) < +\infty,$$

there exists a sub-sequence (not relabeled) and $\bar{m} \in BV(\mathbb{T}^2; \{\pm e_1\})$ such that

$$\int_{(0,1) \times \mathbb{T}^2} |m_k(x) - \bar{m}(x')| \, dx \rightarrow 0 \text{ for } k \rightarrow \infty. \quad (1.15)$$

(ii) **Γ -Convergence:** *The sequence of functionals $(J_k)_{k \in \mathbb{N}}$ Γ -converges towards $J_* : L^1(\mathbb{T}^2; \{\pm e_1\}) \rightarrow \mathbb{R} \cup \{+\infty\}$ given by*

$$J_*(\bar{m}) = \begin{cases} 2 \left(1 - \frac{\lambda}{\lambda_c}\right) \int_{\mathbb{T}^2} |\nabla \bar{m}_1| \, dx - 2 \int_{\mathbb{T}^2} g \bar{m}_1 \, dx, & \text{if } \bar{m} \in BV(\mathbb{T}^2; \{\pm e_1\}), \\ +\infty & \text{otherwise.} \end{cases}$$

This means

(a) **liminf - Inequality:** *Every sequence $(m_k)_{k \in \mathbb{N}}$ in $L^1((0,1) \times \mathbb{T}^2; \mathbb{S}^2)$ that converges towards $\bar{m} \in L^1(\mathbb{T}^2; \{\pm e_1\})$ in the sense of (1.15) satisfies*

$$\liminf_{k \rightarrow \infty} J_k(m_k) \geq J_*(\bar{m}).$$

(b) **Recovery Sequence:** *For every $\bar{m} \in L^1(\mathbb{T}^2, \{\pm e_1\})$ there exists a sequence of magnetizations $(m_k)_{k \in \mathbb{N}}$ in $L^1((0,1) \times \mathbb{T}^2; \mathbb{S}^2)$, which converges towards \bar{m} in the sense of (1.15) and satisfies*

$$\limsup_{k \rightarrow \infty} J_k(m_k) \leq J_*(\bar{m}).$$

Whereas the energy favors single domain states in the subcritical regime, our next theorem shows that the energy leads to pattern formation in the supercritical regime.

Theorem 1.2.2 (Supercritical regime). *Let $h_{\text{ext}} = 0$. There are universal constants $\delta, K > 0$ such that for $Q, \ell, t > 0$ in the regime*

$$Q > 1, \quad t \leq \delta \min \left\{ \sqrt{Q-1}, \frac{1}{\sqrt{Q-1}} \right\} \quad \text{and} \quad \ell \geq K \frac{e^{2\pi t^{-1}\sqrt{Q-1}}}{\sqrt{Q-1}} \quad (1.16)$$

the minimal renormalized energy J in (1.13) scales as

$$-Ct\ell e^{-2\pi t^{-1}\sqrt{Q-1}} \leq \min J(m) \leq -ct\ell e^{-2\pi t^{-1}\sqrt{Q-1}},$$

for some universal constants $0 < c < C$.

Furthermore, profiles achieving the optimal scaling in the regime (1.16) can be characterized as follows.

Proposition 1.2.3. *Let δ, K be as in Theorem 1.2.2, $h_{\text{ext}} = 0$ and let ℓ, t, Q satisfy (1.16). For any $\gamma > 0$ and all $m \in H^1((0, 1) \times \mathbb{T}^2; \mathbb{S}^2)$ which satisfy*

$$J(m) \leq -\gamma t\ell e^{-2\pi t^{-1}\sqrt{Q-1}}, \quad (1.17)$$

we have

$$(i) \quad \int_{\mathbb{T}^2 \times (0,1)} |m - \bar{m}|^2 \, dx \leq c_\gamma t^3 e^{-2\pi t^{-1}\sqrt{Q-1}} \sqrt{Q-1}, \quad (1.18)$$

$$(ii) \quad \int_{\mathbb{T}^2 \times (0,1)} (m_2^2 + m_3^2) \, dx \leq c_\gamma e^{-2\pi t^{-1}\sqrt{Q-1}}, \quad (1.19)$$

$$(iii) \quad c_\gamma \ell e^{-2\pi t^{-1}\sqrt{Q-1}} \sqrt{Q-1} \leq \int_{\mathbb{T}^2} |\nabla' \bar{m}_1| \, dx \leq C_\gamma \ell e^{-2\pi t^{-1}\sqrt{Q-1}} \sqrt{Q-1} \quad (1.20)$$

$$(iv) \quad \int_{\mathbb{T}^2 \times (0,1)} \left(\frac{|\nabla m|^2}{\ell \sqrt{Q-1}} + \ell \sqrt{Q-1} (1 - m_1^2) \right) \, dx - 2 \int_{\mathbb{T}^2} |\nabla \bar{m}_1| \, dx \leq c_\gamma \frac{t}{\sqrt{Q-1}} \int_{\mathbb{T}^2} |\nabla \bar{m}_1| \, dx, \quad (1.21)$$

where $0 < c_\gamma < C_\gamma$ are constants (changing from line to line) which may depend only on γ .

We take a moment to interpret the statements (i)–(iv) in Proposition 1.2.3 above. Item (i) shows that the magnetization is approximately two-dimensional, i.e. independent of the thickness variable. Moreover, since $|m| = 1$, item (ii) means that the magnetization is mostly perpendicular to the film (i.e. $m \approx \pm e_1$). Furthermore, item (iii) is an estimate for the total length of the domain walls in a unit cell. Back in the original, physical variables, this quantity for the unit cell $(0, T) \times (0, L)^2$ is

$$W := L \int_{\mathbb{T}^2} |\nabla \bar{m}_1| \, dx \stackrel{(1.20)}{\sim} \frac{L^2}{l_{\text{ex}}} e^{-\frac{2\pi l_{\text{ex}} \sqrt{Q-1}}{T}} \sqrt{Q-1}. \quad (1.22)$$

We expect that the stray field energy induces a repulsive interaction of (nearest) neighboring domain walls and leads to an approximately equidistant spacing of the walls. In view of (iii), the typical distance of neighboring walls should be

$$S := \frac{\text{length of the film}}{\# \text{ of walls on cross section}} \sim \frac{\ell}{\int_{\mathbb{T}_2} |\nabla' \bar{m}_1| \, dx} l_{\text{ex}} \stackrel{(1.20)}{\sim} \frac{l_{\text{ex}} e^{\frac{2\pi l_{\text{ex}} \sqrt{Q-1}}{T}}}{\sqrt{Q-1}}. \quad (1.23)$$

The exponential dependence of the typical distance between neighboring walls on the inverse thickness in (1.23) was already observed in ansatz based computations in [51] for a two-dimensional sharp interface model. Item (iv) in Proposition 1.2.3 indicates that domain walls approximate Bloch walls of thickness proportional to $\varepsilon L = \frac{l_{\text{ex}}}{\sqrt{Q-1}}$ for which the left hand side of (1.21) is exactly zero. Note that (1.21) also implies that m approximately satisfies the optimal profile ODE in an L^2 -sense

$$\begin{aligned} \int_{(0,1) \times \mathbb{T}^2} \left(\frac{|\nabla m_1|}{\sqrt{\ell \sqrt{Q-1} (1-m_1^2)}} - \sqrt{\ell \sqrt{Q-1} (1-m_1^2)} \right)^2 dx \\ \lesssim \frac{t}{\sqrt{Q-1}} \int_{\mathbb{T}^2} |\nabla m_1| \, dx, \end{aligned}$$

with the convention $\frac{|\nabla m_1|}{\sqrt{1-m_1^2}} = 0$ if $|m_1| = 1$. Finally, we want to mention that the estimate of the in-plane magnetization in item (i) is consistent with the in-plane magnetization of a Bloch wall of length W (see (1.22)) and thickness $\frac{l_{\text{ex}}}{\sqrt{Q-1}}$.

Our third theorem addresses the transition where the cross-over from constant to non-constant global minimizers occurs and which separates the two previously described regimes.

Theorem 1.2.4 (Critical scaling). *Let $h_{\text{ext}} = 0$ and let $\delta > 0$ be as in Theorem 1.2.2. Then the following holds*

- (i) **Cross-over of global minimizers** *There are constants $c, C > 0$ such that for ℓ, t, Q which satisfy*

$$Q > 1, \quad t \leq \delta \min \left\{ \sqrt{Q-1}, \frac{1}{\sqrt{Q-1}} \right\} \quad \text{and} \quad \ell \leq c \frac{e^{2\pi t^{-1}\sqrt{Q-1}}}{\sqrt{Q-1}}$$

the renormalized energy J is non-negative and $m \equiv \pm e_1$ are the only global minimizers, whereas for ℓ, t, Q which satisfy

$$Q > 1, \quad t \leq \delta \min \left\{ \sqrt{Q-1}, \frac{1}{\sqrt{Q-1}} \right\} \quad \text{and} \quad \ell \geq C \frac{e^{2\pi t^{-1}\sqrt{Q-1}}}{\sqrt{Q-1}}$$

the minimal rescaled energy $\min J$ is strictly negative and minimizers cannot be constant.

- (ii) **Γ -convergence** *For $\frac{t \log(\ell\sqrt{Q-1})}{\sqrt{Q-1}} = 2\pi$, J Γ -converges for $\ell\sqrt{Q-1} \rightarrow \infty$ towards*

$$J_*(m) = \begin{cases} 0 & \text{if } m \in L^1(\mathbb{T}^2; \{\pm e_1\}), \\ +\infty & \text{otherwise.} \end{cases}$$

- (iii) **Compactness upon rescaling** *For $C > 0$ and $\ell\sqrt{Q-1} \rightarrow \infty$, sequences with*

$$J(m) \leq \frac{C}{\log(\ell\sqrt{Q-1})}$$

are compact in $L^1((0,1) \times \mathbb{T}^2)$ with a limit of the form $\chi_{(0,1)} \bar{m}$ where $\bar{m} \in BV^1(\mathbb{T}^2; \{\pm e_1\})$.

1.2.2 Results for the simplified energy F

In this section, we will formulate results analogous to the ones in the previous section, but for the reduced energy F . The relation between the full energy E and the reduced two-dimensional energy F was explained heuristically in section 1.1.1 and will be made rigorous in section 1.5. The reason to formulate our results also in terms of F is mainly expositional: We believe that the main ideas are easier to understand when they are not obscured by additional

difficulties arising from the reduction to a two-dimensional model and the stray-field energy approximation.

The behavior of the reduced energy F in the subcritical regime is summarized in the following theorem.

Theorem 1.2.5 (Subcritical regime). *Let $\lambda < \lambda_c := \frac{\pi}{2}$ and $F_{\varepsilon,\lambda}$ as defined in (1.11). Then the following holds*

(i) **Compactness:** *Every sequence $\{m_\varepsilon\}_{\varepsilon>0}$ in $H^1(\mathbb{T}^2; \mathbb{S}^2)$ with*

$$\limsup_{\varepsilon \rightarrow 0} F_{\varepsilon,\lambda}(m_\varepsilon) < +\infty$$

converges in $L^1(\mathbb{T}^2)$ (up to extracting a subsequence) towards a limit in $BV(\mathbb{T}^2; \{\pm e_1\})$.

(ii) **Γ -convergence:** *As $\varepsilon \rightarrow 0$, the sequence of functionals $\{F_{\varepsilon,\lambda}\}_{\varepsilon>0}$ Γ -converges with respect to the $L^1(\mathbb{T}^2)$ -topology towards $F_{*,\lambda}$, given by*

$$F_{*,\lambda}(m) = \begin{cases} \left(1 - \frac{\lambda}{\lambda_c}\right) \int_{\mathbb{T}^2} |\nabla m_1| \, dx & \text{for } m \in BV(\mathbb{T}^2; \{\pm e_1\}), \\ +\infty & \text{otherwise.} \end{cases} \quad (1.24)$$

The next theorem is concerned with the minimal energy and the structure of low energy states in the supercritical regime.

Theorem 1.2.6 (Supercritical regime). *Let $\lambda_c := \frac{\pi}{2}$ and $F_{\varepsilon,\lambda}$ as defined in (1.11). There are constants $\delta < 1 < K$ such that for*

$$0 < \varepsilon < K^{-\frac{\lambda}{\lambda-\lambda_c}} \quad \text{and} \quad \lambda_c < \lambda < \delta |\log \varepsilon|,$$

the minimal energy of $F_{\varepsilon,\lambda}$ satisfies

$$-C \frac{\lambda \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}}}{|\log \varepsilon|} \leq \min F_{\varepsilon,\lambda} \leq -c \frac{\lambda \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}}}{|\log \varepsilon|}$$

for some universal constants $0 < c < C$. Moreover, the profiles achieving the optimal scaling can be characterized as follows. For any $\gamma > 0$ and all $m \in H^1(\mathbb{T}^2; \mathbb{S}^2)$ which satisfy

$$F_{\varepsilon,\lambda}(m) \leq -\gamma \frac{\lambda \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}}}{|\log \varepsilon|},$$

the quantities

$$\begin{aligned} \int_{\mathbb{T}^2} |\nabla m_1| \, dx &\leq \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2} |\nabla m|^2 + \frac{1 - m_1^2}{2\varepsilon} \right) \, dx \\ &\leq \frac{\lambda}{|\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla^{\frac{1}{2}} m_1|^2 \, dx \end{aligned} \quad (1.25)$$

agree to the leading order and scale as $\varepsilon^{\frac{\lambda_c - \lambda}{\lambda}}$, i.e. if A and B are any of the three quantities in (1.25), we have

$$c_\gamma \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}} \leq A \leq C_\gamma \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}} \quad \text{and} \quad |A - B| \leq \tilde{C}_\gamma \frac{\lambda}{|\log \varepsilon|} A, \quad (1.26)$$

for some positive constants c_γ, C_γ and \tilde{C}_γ which depend only on γ .

Under the assumptions of Theorem 1.2.6, statements analogous to (1.18) – (1.21) in Proposition 1.2.3 hold as well, they are simple consequences of the stronger statement (1.26).

The next theorem addresses the structure of minimizers in a neighborhood of the transition.

Theorem 1.2.7 (Critical scaling). *Let $\lambda_c := \frac{\pi}{2}$ and $F_{\varepsilon, \lambda_c}$ as defined in (1.11). Then the following holds*

- (i) **Cross-over of global minimizers:** *There are two constants $0 < \beta_1 < 1 < \beta_2$ such that for*

$$\lambda \leq \lambda_-(\varepsilon) := \lambda_c \left(1 - \frac{|\log \beta_1|}{|\log \varepsilon|} \right) \quad (1.27)$$

the minimal energy $\min F_{\varepsilon, \lambda}$ is zero and only attained by the constant configurations $m \equiv \pm e_1$, whereas for

$$\lambda \geq \lambda_+(\varepsilon) := \lambda_c \left(1 + \frac{|\log \beta_2|}{|\log \varepsilon|} \right)$$

the minimal energy is strictly negative and minimizers cannot be constant.

- (ii) **Γ -convergence:** *As $\varepsilon \rightarrow 0$, the sequence of functionals $\{F_{\varepsilon, \lambda_c}\}_{\varepsilon > 0}$ Γ -converges with respect to the $L^1(\mathbb{T}^2)$ -topology towards F_{*, λ_c} , given by*

$$F_{*, \lambda_c}(m) = \begin{cases} 0, & \text{if } m \in L^1(\mathbb{T}^2; \{\pm e_1\}) \\ +\infty & \text{otherwise,} \end{cases} \quad (1.28)$$

(iii) **Lack of compactness:** There is a sequence $\{m_\varepsilon\}_{\varepsilon>0}$ in $H^1(\mathbb{T}^2; \mathbb{S}^2)$ with

$$\limsup_{\varepsilon \rightarrow 0} F_{\varepsilon, \lambda_c}(m_\varepsilon) \rightarrow 0$$

which is not precompact in $L^1(\mathbb{T}^2)$.

(iv) **Compactness upon rescaling:** For every $C > 0$, every sequence $\{m_\varepsilon\}_{\varepsilon>0}$ with

$$F_{\varepsilon, \lambda_c}(m_\varepsilon) \leq C |\log \varepsilon|^{-1}$$

converges in $L^1(\mathbb{T}^2)$ (up to extracting a subsequence) to a limit in $BV(\mathbb{T}^2; \{\pm e_1\})$.

Theorem 1.2.7 suggests that $|\log \varepsilon| F_{\varepsilon, \lambda_c}$ is the appropriate rescaling for the critical case. Unfortunately, it seems not possible to obtain the Γ -limit of $|\log \varepsilon| F_{\varepsilon, \lambda_c}$ with our $H^{\frac{1}{2}}$ -estimate (1.29) of the following section. However, we will derive a Γ -limit for a related sharp interface model in chapter 2.

We illustrate our results in a phase diagram (Figure 1.3). It is not difficult to see that for each $0 < \varepsilon < 1$ there is a sharp threshold value $\lambda = \lambda_c(\varepsilon) > 0$ at which a transition from monodomain ($m \equiv +e_1$ or $m \equiv -e_1$) to multidomain ($m \not\equiv \text{const}$) states as global energy minimizers occurs. Moreover, $\varepsilon \mapsto \lambda_c(\varepsilon)$ is locally Lipschitz-continuous on $(0, 1)$ (for the reader's convenience, a proof of this fact is presented in Lemma A.1 in the appendix).

While we do not know the precise value of $\lambda_c(\varepsilon)$ for $\varepsilon > 0$, we show in Theorem 1.2.7 that $\lambda_-(\varepsilon) \leq \lambda_c(\varepsilon) \leq \lambda_+(\varepsilon)$ and $\lim_{\varepsilon \rightarrow 0} \lambda_c(\varepsilon) = \frac{\pi}{2}$, i.e. the definition above agrees with $\lambda_c := \lambda_c(0) = \frac{\pi}{2}$. Furthermore, global minimizers $m_{(\varepsilon, \lambda)}$ of $F_{\varepsilon, \lambda}$ with (ε, λ) between the red (dashed) curves of the form $\lambda(\varepsilon) = \lambda_c + \gamma |\log \varepsilon|^{-1}$ satisfy a uniform bound of the form $c \leq \int_{\mathbb{T}^2} |\nabla m_{(\varepsilon, \lambda), 1}| \, dx \leq C$, with constants $C > c > 0$ depending only on the values of $\gamma > 0$ for these curves.

1.3 A bound on the homogeneous $H^{\frac{1}{2}}$ -norm

Since all three terms in F contribute in highest order to the limit, it is important to estimate the negative term $\int_{\mathbb{T}^2} |\nabla^{\frac{1}{2}} m_1|^2 \, dx$ with precise leading order constant. In this section we will establish an upper bound for the homogeneous $H^{\frac{1}{2}}$ -norm which is the key ingredient for the lower bounds (recall that the $H^{\frac{1}{2}}$ -term occurs in the energy with a negative sign).

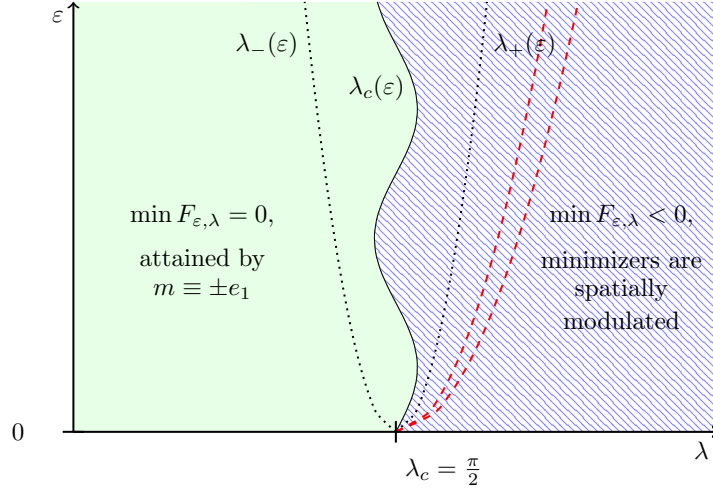


Figure 1.3: Sketch of the phase diagram for minimizers of $F_{\epsilon, \lambda}$ in terms of $\lambda > 0$ and $0 < \epsilon \ll 1$.

We will prove the following

Lemma 1.3.1. *There is a universal constant $c_* \geq 1$ such that for every $f \in C^\infty(\mathbb{T}^2)$ and every $\epsilon > 0$ we have*

$$\begin{aligned} \int_{\mathbb{T}^2} |\nabla^{\frac{1}{2}} f|^2 \, dx &\leq \frac{\epsilon}{2} \int_{\mathbb{T}^2} |\nabla f|^2 \, dx \\ &+ \frac{2}{\pi} \log \left(c_* \max \left\{ 1, \min \left\{ \frac{\|f\|_\infty}{\epsilon \int_{\mathbb{T}^2} |\nabla f| \, dx}, \frac{1}{\epsilon} \right\} \right\} \right) \|f\|_\infty \int_{\mathbb{T}^2} |\nabla f| \, dx. \end{aligned} \quad (1.29)$$

In Lemma 1.3.1, we improve an inequality established in [26]. Expressed in our setting, the inequality proved in [26] asserts that for every $\delta > 0$ there exists $M_\delta \gg 1$ such that for all $\epsilon \leq R$ and all $f : \mathbb{T}^2 \rightarrow \mathbb{R}$, we have

$$\begin{aligned} \sum_{k \in 2\pi\mathbb{Z}^2} \min \left\{ \frac{1}{\epsilon}, |k|, R|k|^2 \right\} |\widehat{f}_k|^2 \\ \leq (1 + \delta) \frac{2}{\pi} \log \left(\frac{2M_\delta R}{\epsilon} \right) \|f\|_\infty \int_{\mathbb{T}^2} |\nabla f| \, dx. \end{aligned} \quad (1.30)$$

Note that (1.29) implies a similar estimate

$$\int_{\mathbb{T}^2} |\nabla^{\frac{1}{2}} f|^2 \, dx \leq \frac{\epsilon}{2} \int_{\mathbb{T}^2} |\nabla f|^2 \, dx + \frac{2}{\pi} \log(c_*/\epsilon) \|f\|_\infty \int_{\mathbb{T}^2} |\nabla f| \, dx \quad (1.31)$$

for all $\epsilon \leq 1$, which is weaker than (1.29). Estimate (1.30) is an inequality for a regularized $\dot{H}^{\frac{1}{2}}$ -norm, whereas (1.31) estimates the full $\dot{H}^{\frac{1}{2}}$ -norm, but needs

an additional \dot{H}^1 -term. It ceases to be optimal for functions which oscillate significantly. Indeed, let $\alpha \in (0, 1)$ and consider functions f with

$$\int_{\mathbb{T}^2} |\nabla f| \, dx \gtrsim \varepsilon^{-\alpha} \|f\|_{\infty}. \quad (1.32)$$

Then the second term in (1.29) is smaller than the second term in (1.31) by a factor of $(1 - \alpha)$ for all f which satisfy (1.32). Asymptotic optimality in the case of strong oscillation is crucial to obtain the results on the supercritical regime.

The proof of Lemma 1.3.1 uses similar ideas as in [26] and is based on a separate treatment of distinct scales. However, our proof does not involve any Fourier Analysis.

Lemma 1.3.1. We will show that the following estimates hold for all $f \in C^\infty(\mathbb{T}^2)$ and all $0 < r \leq R$:

$$\int_{\mathbb{T}^2} \int_{B_r} \frac{|f(x+z) - f(x)|^2}{|z|^3} \, dz \, dx \leq \pi r \int_{\mathbb{T}^2} |\nabla f|^2 \, dx, \quad (1.33)$$

$$\begin{aligned} \int_{\mathbb{T}^2} \int_{B_R \setminus B_r} \frac{|f(x+z) - f(x)|^2}{|z|^3} \, dz \, dx \\ \leq 8 \log(R/r) \|f\|_{\infty} \int_{\mathbb{T}^2} |\nabla f| \, dx, \end{aligned} \quad (1.34)$$

$$\begin{aligned} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2 \setminus B_R} \frac{|f(x+z) - f(x)|^2}{|z|^3} \, dz \, dx \\ \leq \frac{2\pi \|f\|_{\infty}}{R} \min \left\{ 4\|f\|_{\infty}, \int_{\mathbb{T}^2} |\nabla f| \, dx \right\}. \end{aligned} \quad (1.35)$$

The claim of the Lemma will follow by adding (1.33) – (1.35) and a suitable choice of r and R . Before we start with the proofs of estimates (1.33) – (1.35), we first record an auxiliary inequality for further use. By the Fundamental Theorem of Calculus, Jensen's inequality and Fubini's theorem we get

$$\begin{aligned} \int_{\mathbb{T}^2} |f(x+z) - f(x)|^p \, dx &= \int_{\mathbb{T}^2} \left| \int_0^1 \nabla f(x+sz) \cdot z \, ds \right|^p \, dx \\ &\leq \int_0^1 \int_{\mathbb{T}^2} |\nabla f(x+sz) \cdot z|^p \, dx \, ds \leq \int_{\mathbb{T}^2} |\nabla f(x) \cdot z|^p \, dx \end{aligned} \quad (1.36)$$

for all $z \in \mathbb{R}^2$ and all $1 \leq p < \infty$. In order to prove (1.33), we use Fubini's Theorem and apply (1.36) with $p = 2$ to get

$$\int_{\mathbb{T}^2} \int_{B_r} \frac{|f(x+z) - f(x)|^2}{|z|^3} \, dz \, dx \stackrel{(1.36)}{\leq} \int_{B_r} \int_{\mathbb{T}^2} \frac{|\nabla f(x) \cdot z|^2}{|z|^3} \, dx \, dz. \quad (1.37)$$

We apply Fubini's Theorem again and evaluate the integral with respect to z in polar coordinates

$$\begin{aligned} \int_{B_r} \int_{\mathbb{T}^2} \frac{|\nabla f(x) \cdot z|^2}{|z|^3} dx dz &= \left(\int_0^r \int_0^{2\pi} \cos^2 \varphi d\varphi d\rho \right) \left(\int_{\mathbb{T}^2} |\nabla f(x)|^2 dx \right) \\ &= \pi r \int_{\mathbb{T}^2} |\nabla f|^2 dx. \end{aligned} \quad (1.38)$$

Together, (1.37) and (1.38) yield the first estimate (1.33).

For the estimate involving intermediate distances (1.34), we use Fubini's Theorem (twice) and (1.36) with $p = 1$ to conclude

$$\begin{aligned} \int_{\mathbb{T}^2} \int_{B_R \setminus B_r} \frac{|f(x+z) - f(x)|^2}{|z|^3} dz dx \\ \stackrel{(1.36)}{\leq} 2\|f\|_\infty \int_{\mathbb{T}^2} \int_{B_R \setminus B_r} \frac{|\nabla f(x) \cdot z|}{|z|^3} dz dx. \end{aligned} \quad (1.39)$$

As in the proof of (1.30) in [26], we evaluate the inner integral in polar coordinates

$$\begin{aligned} \int_{B_R \setminus B_r} \frac{|\nabla f(x) \cdot z|}{|z|^3} dz &= \int_r^R \int_0^{2\pi} \frac{|\nabla f(x)| |\cos \varphi|}{\rho} d\varphi d\rho \\ &= 4 \log \left(\frac{R}{r} \right) |\nabla f(x)|. \end{aligned} \quad (1.40)$$

Inserting (1.40) into (1.39) yields the claim (1.34).

In order to prove (1.35), we first show that for all $z \in \mathbb{R}^2$ we have

$$\int_{\mathbb{T}^2} |f(x+z) - f(x)| dx \leq \min \left\{ 2\|f\|_\infty, \frac{1}{2} \int_{\mathbb{T}^2} |\nabla f| dx \right\}. \quad (1.41)$$

Indeed, the upper bound of $2\|f\|_\infty$ in (1.41) is trivial. Furthermore, since f is periodic, it is sufficient to show the second upper bound in (1.41) only for $z \in (-\frac{1}{2}, \frac{1}{2})^2$. Thus the second bound in (1.41) follows from (1.36) with $p = 1$

$$\int_{\mathbb{T}^2} |f(x+z) - f(x)| dx \stackrel{(1.36)}{\leq} \int_{\mathbb{T}^2} |\nabla f(x) \cdot z| dx \leq \frac{1}{2} \int_{\mathbb{T}^2} |\nabla f(x)| dx$$

so that the proof of (1.41) is complete. With (1.41) at hand, estimate (1.35) now follows by direct integration

$$\begin{aligned} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2 \setminus B_R} \frac{|f(x+z) - f(x)|^2}{|z|^3} dz dx \\ \leq 2\|f\|_{L^\infty} \int_{\mathbb{R}^2 \setminus B_R} \int_{\mathbb{T}^2} \frac{|f(x+z) - f(x)|}{|z|^3} dx dz \\ \stackrel{(1.41)}{\leq} \frac{2\pi\|f\|_\infty}{R} \min \left\{ 4\|f\|_\infty, \int_{\mathbb{T}^2} |\nabla f| dx \right\}. \end{aligned}$$

It remains to prove (1.29), for which we use the real-space representation of the homogeneous $H^{\frac{1}{2}}$ -norm

$$\int_{\mathbb{T}^2} |\nabla^{\frac{1}{2}} f|^2 dx = \frac{1}{4\pi} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \frac{|f(x+z) - f(x)|^2}{|z|^3} dz dx. \quad (1.42)$$

A proof of (1.42) is given in the appendix for completeness of the presentation. Without loss of generality, we may assume that f is not equal to a constant in \mathbb{T}^2 . Adding (1.33) – (1.35) to estimate the right hand side of (1.29), we therefore get

$$\begin{aligned} \int_{\mathbb{T}^2} |\nabla^{\frac{1}{2}} f|^2 dx &\leq \frac{r}{4} \int_{\mathbb{T}^2} |\nabla f|^2 dx \\ &+ \left(\frac{2}{\pi} \log \left(\frac{R}{r} \right) + \frac{1}{2R} \min \left\{ \frac{4\|f\|_{\infty}}{\int_{\mathbb{T}^2} |\nabla f| dx}, 1 \right\} \right) \|f\|_{\infty} \int_{\mathbb{T}^2} |\nabla f| dx. \end{aligned} \quad (1.43)$$

For $r = 2\varepsilon$ and $R = \max \left\{ 2\varepsilon, \min \left\{ \frac{4\|f\|_{\infty}}{\int_{\mathbb{T}^2} |\nabla f| dx}, 1 \right\} \right\}$ the claim (1.29) now follows from (1.43). \square

1.4 Proofs for the reduced energy F

In this section we give the proofs of the theorems involving the reduced energy F . The proof of Theorem 1.2.5 is a direct consequence of Lemma 1.4.1 and Lemma 1.4.3. Similarly the proof of Theorem 1.2.6 follows immediately from Lemma 1.4.4 and Lemma 1.4.5. Finally, the proof of Theorem 1.2.7 is presented at the end of this section.

1.4.1 Proof of Theorem 1.2.5

Lemma 1.4.1 (Lower bound and compactness in the subcritical regime). *Let $\lambda < \lambda_c := \frac{\pi}{2}$ and $F_{\varepsilon, \lambda}$ as defined in (1.11). Then every sequence $\{m_{\varepsilon}\}_{\varepsilon > 0}$ in $H^1(\mathbb{T}^2; \mathbb{S}^2)$ with*

$$\limsup_{\varepsilon \rightarrow 0} F_{\varepsilon, \lambda}(m_{\varepsilon}) < +\infty$$

converges in $L^1(\mathbb{T}^2; \mathbb{R}^3)$ (up to extracting a subsequence) towards a limit in $BV(\mathbb{T}^2; \{\pm e_1\})$. Furthermore, for every sequence $\{m_{\varepsilon}\}_{\varepsilon > 0}$ in $L^1(\mathbb{T}^2; \mathbb{S}^2)$ with

$m_\varepsilon \rightarrow m$ for some m in $L^1(\mathbb{T}^2; \mathbb{R}^3)$ we have

$$\liminf_{\varepsilon \rightarrow 0} F_{\varepsilon, \lambda}(m_\varepsilon) \geq \begin{cases} \left(1 - \frac{\lambda}{\lambda_c}\right) \int_{\mathbb{T}^2} |\nabla m_1| \, dx, & \text{if } m \in BV(\mathbb{T}^2; \{\pm e_1\}), \\ +\infty & \text{otherwise.} \end{cases} \quad (1.44)$$

Lemma 1.4.1. We first show that for sufficiently small $\varepsilon > 0$ we have

$$F_{\varepsilon, \lambda}(m) \geq \left(1 - \frac{\lambda |\log c\varepsilon|}{\lambda_c |\log \varepsilon|}\right) \int_{\mathbb{T}^2} |\nabla m_1| \, dx \quad (1.45)$$

for all $m \in H^1(\mathbb{T}^2; \mathbb{S}^2)$, where $c > 0$ is a universal constant. Indeed, for $\lambda < \lambda_c$ we expect $\int_{\mathbb{T}^2} |\nabla m_1| \, dx$ to be small and hence it is sufficient to use Lemma 1.3.1 for m_1 in the weaker form (1.31). Recalling that $\|m_1\|_\infty \leq 1$ and $\lambda_c = \frac{\pi}{2}$, we get

$$\begin{aligned} \frac{\lambda}{|\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla^{\frac{1}{2}} m_1|^2 \, dx &\stackrel{(1.31)}{\leq} \frac{\lambda}{|\log \varepsilon|} \int_{\mathbb{T}^2} \frac{\varepsilon}{2} |\nabla m_1|^2 \, dx \\ &\quad + \frac{\lambda \log(c_*/\varepsilon)}{\lambda_c |\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla m_1| \, dx. \end{aligned} \quad (1.46)$$

We also use the constraint $|m| = 1$ in the form of the well-known estimate

$$|\nabla m_1| \stackrel{(A.7)}{\leq} \frac{\varepsilon}{2} |\nabla m|^2 + \frac{1}{2\varepsilon} (1 - m_1^2). \quad (1.47)$$

which is obtained by differentiating $|m|^2 = 1$ and applying Young's inequality (see (A.7) in the Appendix for a proof). Now the claimed lower bound (1.45) follows from (1.46) and (1.47):

$$\begin{aligned} F_{\varepsilon, \lambda}(m) &= \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2} |\nabla m|^2 + \frac{1}{2\varepsilon} (1 - m_1^2) \right) \, dx - \frac{\lambda}{|\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla^{\frac{1}{2}} m_1|^2 \, dx \\ &\stackrel{(1.46)}{\geq} \left(1 - \frac{\lambda}{|\log \varepsilon|}\right) \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2} |\nabla m|^2 + \frac{1}{2\varepsilon} (1 - m_1^2) \right) \, dx \\ &\quad - \frac{\lambda \log(c_*/\varepsilon)}{\lambda_c |\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla m_1| \, dx \\ &\stackrel{(1.47)}{\geq} \left(1 - \frac{\lambda \log(e^{\lambda_c} c_*/\varepsilon)}{\lambda_c |\log \varepsilon|}\right) \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2} |\nabla m|^2 + \frac{1}{2\varepsilon} (1 - m_1^2) \right) \, dx \\ &\stackrel{(1.47)}{\geq} \left(1 - \frac{\lambda \log(e^{\lambda_c} c_*/\varepsilon)}{\lambda_c |\log \varepsilon|}\right) \int_{\mathbb{T}^2} |\nabla m_1| \, dx. \end{aligned} \quad (1.48)$$

Let $\{m_\varepsilon\}_{\varepsilon>0}$ be a sequence in $H^1(\mathbb{T}^2; \mathbb{S}^2)$ with bounded energy $\limsup_{\varepsilon \rightarrow 0} F_{\varepsilon, \lambda}(m_\varepsilon) < +\infty$. From the penultimate line in (1.48), $|m_\varepsilon| = 1$ and $\lambda < \lambda_c$ we obtain

$$0 = \limsup_{\varepsilon \rightarrow 0} \varepsilon F_{\varepsilon, \lambda}(m_\varepsilon) \stackrel{(1.48)}{\geq} \frac{1}{2} \left(1 - \frac{\lambda}{\lambda_c}\right) \limsup_{\varepsilon \rightarrow 0} \int_{\mathbb{T}^2} (m_{\varepsilon, 2}^2 + m_{\varepsilon, 3}^2) \, dx,$$

implying that the last two components $m_{\varepsilon, 2}$ and $m_{\varepsilon, 3}$ converge to zero in $L^2(\mathbb{T}^2)$ as $\varepsilon \rightarrow 0$. Moreover, (1.48) yields a uniform bound for $m_{\varepsilon, 1}$ in BV , which by compactness of $BV(\mathbb{T}^2)$ in $L^1(\mathbb{T}^2)$ implies the existence of a convergent subsequence. Passing to another subsequence, we may assume that m_ε converges pointwise almost everywhere. Since $|m_\varepsilon| = 1$, we obtain $m = \pm e_1$ almost everywhere.

For the liminf inequality (1.44), we may assume without loss of generality that $\liminf_{\varepsilon \rightarrow 0} F_{\varepsilon, \lambda}(m_\varepsilon) < +\infty$. But then there is a subsequence (not relabelled) such that $\limsup_{\varepsilon \rightarrow 0} F_{\varepsilon, \lambda}(m_\varepsilon) < +\infty$ and by the compactness result and uniqueness of the limit we have $m \in BV(\mathbb{T}^2; \{\pm e_1\})$. Now the liminf inequality follows directly from (1.45), the fact that $\lim_{\varepsilon \rightarrow 0} \frac{\lambda |\log c\varepsilon|}{\lambda_c |\log \varepsilon|} = \frac{\lambda}{\lambda_c} < 1$ and lower semi-continuity of the BV -seminorm. \square

Before we begin with the construction of the upper bound, we define a family of asymptotically optimal profiles and record some of their properties (see Fig. 1.4).

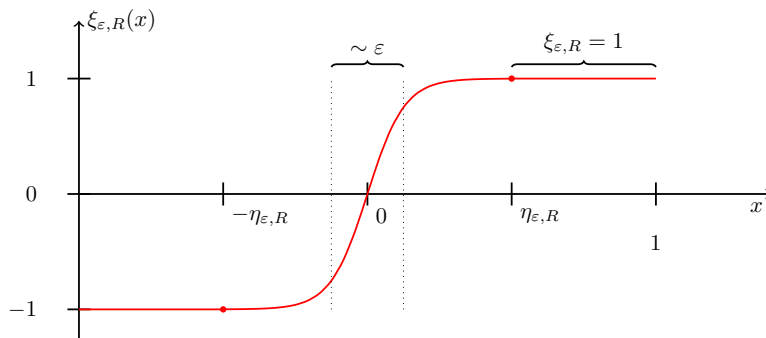


Figure 1.4: Family of asymptotically optimal profiles $\xi_{\varepsilon, R}$

Lemma 1.4.2 (Estimates for a family of asymptotically optimal profiles). *For $R \in (0, +\infty]$ and $\varepsilon > 0$, let $\xi_{\varepsilon, R} : \mathbb{R} \rightarrow [-1, 1]$ be the unique solution to the*

initial value problem

$$\xi_{\varepsilon,R}(0) = 0 \quad \text{and} \quad \xi'_{\varepsilon,R} = \frac{1}{\varepsilon}(1 - \xi_{\varepsilon,R}^2)^{\frac{1}{2}} \left(1 - \xi_{\varepsilon,R}^2 + \left(\frac{\pi\varepsilon}{2R}\right)^2\right)^{\frac{1}{2}}. \quad (1.49)$$

Then $\xi_{\varepsilon,R}$ is non-decreasing and satisfies

$$\xi_{\varepsilon,R}(x) = -\xi_{\varepsilon,R}(-x) \quad \text{and} \quad |\xi_{\varepsilon,R}(x) - \text{sign}(x)| \leq 2e^{-2|x|/\varepsilon}. \quad (1.50)$$

Moreover, $\xi_{\varepsilon,R}(x) = 1$ if $x \geq \eta_{\varepsilon,R}$ and $\xi_{\varepsilon,R}(x) = -1$ if $x \leq -\eta_{\varepsilon,R}$, for some $\eta_{\varepsilon,R} \in (0, R]$. The contribution to the local part of the energy may be estimated as

$$\frac{1}{2} \int_{-\eta_{\varepsilon,R}}^{\eta_{\varepsilon,R}} \left(\frac{\varepsilon |\xi'_{\varepsilon,R}|^2}{1 - \xi_{\varepsilon,R}^2} + \frac{1 - \xi_{\varepsilon,R}^2}{\varepsilon} \right) dx \leq 2 + \frac{\pi^2 \varepsilon}{4R}. \quad (1.51)$$

Lastly, there is a universal constant $c > 0$ such that

$$\int_{-X}^X \int_{-X}^X \frac{|\xi_{\varepsilon,R}(x) - \xi_{\varepsilon,R}(y)|^2}{|x - y|^2} dx dy \geq 8 \log(cX/\varepsilon) \quad \text{for } X \geq 2\varepsilon. \quad (1.52)$$

Proof. The existence, uniqueness and monotonicity of $\xi_{\varepsilon,R}$ follows by direct integration. In particular, for $R < +\infty$, there exists a unique real number $\eta_{\varepsilon,R} > 0$, such that the solution of (1.49) satisfies $\xi_{\varepsilon,R}(s) \in (-1, 1)$ for $s \in (-\eta_{\varepsilon,R}, \eta_{\varepsilon,R})$ and $\xi_{\varepsilon,R}(\pm\eta_{\varepsilon,R}) = \pm 1$. For $R = +\infty$, we have $\xi_{\varepsilon,\infty} = \tanh(\cdot/\varepsilon)$ and the claim follows for $\eta_{\varepsilon,\infty} = +\infty$. Estimate (1.50) follows immediately from $\xi_{\varepsilon,\infty} \leq \xi_{\varepsilon,R} \leq 1$ for $x \geq 0$.

We will now show that $\eta_{\varepsilon,R} \leq R$ holds. Indeed, since $\xi_{\varepsilon,R}$ is strictly monotone on $(-\eta_{\varepsilon,R}, \eta_{\varepsilon,R})$, the inverse function theorem yields

$$\begin{aligned} \eta_{\varepsilon,R} &= \lim_{s \rightarrow 1^-} \xi_{\varepsilon,R}^{-1}(s) = \int_0^1 (\xi_{\varepsilon,R}^{-1})'(s) ds \\ &= \int_0^1 \frac{\varepsilon}{\sqrt{(1-s^2)(1-s^2 + \frac{\pi^2 \varepsilon^2}{4R^2})}} ds \leq \int_0^1 \frac{2R}{\pi \sqrt{1-s^2}} ds = R. \end{aligned} \quad (1.53)$$

We turn to the proof of (1.51). By (1.49), we have

$$\begin{aligned} &\frac{\varepsilon |\xi'_{\varepsilon,R}|^2}{1 - \xi_{\varepsilon,R}^2} + \frac{1 - \xi_{\varepsilon,R}^2}{\varepsilon} \stackrel{(1.49)}{=} 2\xi'_{\varepsilon,R} + \frac{1}{\varepsilon} \left(\sqrt{1 - \xi_{\varepsilon,R}^2 + \left(\frac{\pi\varepsilon}{2R}\right)^2} - \sqrt{1 - \xi_{\varepsilon,R}^2} \right)^2 \\ &= 2\xi'_{\varepsilon,R} + \frac{1}{\varepsilon} \left(\int_0^{\frac{\pi\varepsilon}{2R}} \frac{s}{\sqrt{1 - \xi_{\varepsilon,R}^2 + s^2}} ds \right)^2 \leq 2\xi'_{\varepsilon,R} + \frac{\pi^2 \varepsilon}{4R^2} \end{aligned} \quad (1.54)$$

and thus (1.51) follows from (1.54) by integration.

It remains to prove (1.52). By symmetry of $\xi_{\varepsilon,R}$ we have

$$\begin{aligned} & \int_{-X}^X \int_{\{\varepsilon \leq |z| \leq X\} \cap \{|x+z| \leq X\}} \frac{|\xi_{\varepsilon,R}(x+z) - \xi_{\varepsilon,R}(x)|^2}{|z|^2} dz dx \\ &= 2 \int_{-X}^0 \int_{\{\varepsilon \leq |z| \leq X\} \cap \{|x+z| \leq X\}} \frac{|\xi_{\varepsilon,R}(x+z) - \xi_{\varepsilon,R}(x)|^2}{|z|^2} dz dx \end{aligned} \quad (1.55)$$

As it turns out, it is sufficient to restrict the integral to a set where $|\xi_{\varepsilon,R}(x+z) - \xi_{\varepsilon,R}(x)| \gtrsim 1$ to obtain the correct leading order behavior

$$\begin{aligned} & \int_{-X}^0 \int_{\{\varepsilon \leq |z| \leq X\} \cap \{|x+z| \leq X\}} \frac{|\xi_{\varepsilon,R}(x+z) - \xi_{\varepsilon,R}(x)|^2}{|z|^2} dz dx \\ & \geq \int_{-X}^0 \int_{\varepsilon}^{x+X} \frac{|\xi_{\varepsilon,R}(y) - \xi_{\varepsilon,R}(x)|^2}{|y-x|^2} dy dx. \end{aligned}$$

Since $|1 - \xi_{\varepsilon,R}|$ decays exponentially with rate $1/\varepsilon$, we split the integral into the leading order and a lower order correction

$$\begin{aligned} & \int_{-X}^0 \int_{\varepsilon}^{x+X} \frac{|\xi_{\varepsilon,R}(y) - \xi_{\varepsilon,R}(x)|^2}{|y-x|^2} dy dx = \int_{-X}^0 \int_{\varepsilon}^{x+X} \frac{4}{|y-x|^2} dy dx \\ & - \int_{-X}^0 \int_{\varepsilon}^{x+X} \frac{4 - |\xi_{\varepsilon,R}(y) - \xi_{\varepsilon,R}(x)|^2}{|y-x|^2} dy dx. \end{aligned} \quad (1.56)$$

The first term on the right hand side of (1.56) yields

$$\int_{-X}^0 \int_{\varepsilon}^{x+X} \frac{1}{|y-x|^2} dy dx = \log \left(\frac{\varepsilon + X}{\varepsilon} \right) - 1.$$

Thus, it is sufficient to show that the second term on the right hand side of (1.56) is bounded independently of ε . Indeed, using the exponential decay of $|1 - \xi_{\varepsilon,R}|$, we get

$$\begin{aligned} & \int_{-X}^0 \int_{\varepsilon}^{x+X} \frac{4 - |\xi_{\varepsilon,R}(y) - \xi_{\varepsilon,R}(x)|^2}{|y-x|^2} dy dx \\ & \lesssim \int_0^{\infty} \int_1^{\infty} \frac{e^{-2x} + e^{-2y}}{|x+y|^2} dx dy \lesssim 1. \end{aligned} \quad (1.57)$$

Together, (1.55) – (1.57) yield the claim (1.52). \square

For the special case $\lambda = 0$, the Γ -convergence and in particular the construction of a recovery sequence is a classical result, relying on the optimal one-dimensional transition profiles to smooth out the jump discontinuity in the

limit configuration [5]. As it turns out, this construction also works for $\lambda > 0$, where $F_{\varepsilon,\lambda}$ is nonlocal. We will use a construction based on the nearly optimal profile $\xi_{\varepsilon,R}$ from Lemma 1.4.2. As the calculations for the local part of the energy are well-known, our focus is on the contribution of the homogeneous $H^{\frac{1}{2}}$ -norm. Recall that we need to prove a lower bound for the $H^{\frac{1}{2}}$ -norm in order to obtain an upper bound for F .

Lemma 1.4.3 (Construction of a recovery sequence in the subcritical and critical regime). *Let $\lambda \leq \lambda_c$ and $m \in L^1(\mathbb{T}^2; \mathbb{S}^2)$. Then there is a sequence $\{m_\varepsilon\}_{\varepsilon>0}$ in $H^1(\mathbb{T}^2; \mathbb{S}^2)$ with*

$$\limsup_{\varepsilon \rightarrow 0} F_{\varepsilon,\lambda}(m_\varepsilon) \leq F_{*,\lambda}(m),$$

where $F_{\varepsilon,\lambda}$ is given by (1.11), and $F_{*,\lambda}$ is given by (1.24) for $\lambda < \lambda_c$ or (1.28) for $\lambda = \lambda_c$, respectively.

Lemma 1.4.3. It is sufficient to prove the limsup inequality under the additional assumption that $m = (\chi_A - \chi_{\mathbb{T}^2 \setminus A})e_1$ for a set $A \subset \mathbb{T}^2$ with smooth boundary. By standard density results (see, e.g., [67, Prop. 12.20]) and a diagonal argument, the limsup inequality then extends to arbitrary $A \subset \mathbb{T}^2$ with finite perimeter for $\lambda < \lambda_c$ or to measurable $A \subset \mathbb{T}^2$ for the $\lambda = \lambda_c$ case. Since $F_{*,\lambda}(m) = +\infty$ for $m \notin BV(\mathbb{T}^2, \{\pm e_1\})$ when $\lambda < \lambda_c$ or for $m \notin L^1(\mathbb{T}^2, \{\pm e_1\})$ when $\lambda = \lambda_c$, this yields the claim.

Our strategy is to adapt the optimal profiles $\xi_{\varepsilon,R}$ from Lemma 1.4.2 to the two-dimensional setting by means of the signed distance function d , given by $d(x) := \text{dist}(x, A^c) - \text{dist}(x, A)$. Without loss of generality, we may assume $0 < |A| < 1$ (otherwise take $m_\varepsilon \equiv \pm e_1$). To fix the notation, let $\nu : \partial A \rightarrow \mathbb{R}^2$ denote the smooth inward normal to A and $\tau : \partial A \rightarrow \mathbb{R}^2$, $\tau = \nu^\perp$ denote a smooth tangent vector field to ∂A obtained by a counter-clockwise 90° rotation of ν . As ∂A is assumed to be smooth, there exists a tubular neighborhood $(\partial A)_R = \bigcup_{x \in \partial A} B_R(x) \subset \mathbb{T}^2$ for some $R > 0$ such that the projection $p : (\partial A)_R \rightarrow \partial A$, $p(x) := \operatorname{argmin}_{y \in \partial A} |x - y|$ is single-valued and hence well-defined. Furthermore, the projection p and the signed distance function d are smooth on $(\partial A)_R$ and the identity

$$x = p(x) + d(x)\nu(p(x))$$

holds for all $x \in (\partial A)_R$, see, e.g., [37, Lemma 14.16].

With the necessary notation at hand, we define the recovery sequence by

$$m_\varepsilon(x) = \xi_{\varepsilon,R}(d(x))e_1 + \sqrt{1 - \xi_{\varepsilon,R}^2(d(x))} \tau(p(x)). \quad (1.58)$$

Recall that $\eta_{\varepsilon,R} \leq R$, (see (1.53)) and hence the function m_ε is Lipschitz continuous and piecewise smooth.

It is easy to see that $m_\varepsilon \rightarrow m$ in $L^1(\mathbb{T}^2)$, and for the sake of completeness, we briefly mention how to compute the contribution of the local energy terms. Since $\tau \perp e_1$, $(\tau \circ p) \cdot \nabla(\tau \circ p) = 0$ and $|\nabla d| = 1$ almost everywhere, the squared gradient of m_ε can be estimated by

$$|\nabla m_\varepsilon|^2 = \frac{|\xi'_{\varepsilon,R}(d)|^2}{1 - \xi_{\varepsilon,R}^2(d)} + (1 - \xi_{\varepsilon,R}^2(d)) |\nabla(\tau \circ p)|^2 \leq \frac{|\xi'_{\varepsilon,R}(d)|^2}{1 - \xi_{\varepsilon,R}^2(d)} + C_A, \quad (1.59)$$

where $C_A > 0$ is a constant that depends only on A for all $R \leq R_A$, where $R_A > 0$ depends only on A . In the following, C_A may change from line to line. We next employ the co-area formula, to reduce to the one-dimensional case:

$$\begin{aligned} & \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2} |\nabla m_\varepsilon|^2 + \frac{1}{2\varepsilon} (1 - m_{\varepsilon,1}^2) \right) dx \\ & \stackrel{(1.59)}{\leq} \int_{(\partial A)_{\eta_{\varepsilon,R}}} \left(\frac{\varepsilon |\xi'_{\varepsilon,R}(d)|^2}{2(1 - \xi_{\varepsilon,R}^2(d))} + \frac{1}{2\varepsilon} (1 - \xi_{\varepsilon,R}^2(d)) \right) dx + \varepsilon C_A \\ & \leq \int_{-\eta_{\varepsilon,R}}^{\eta_{\varepsilon,R}} \left(\frac{\varepsilon |\xi'_{\varepsilon,R}(s)|^2}{2(1 - \xi_{\varepsilon,R}^2(s))} + \frac{1}{2\varepsilon} (1 - \xi_{\varepsilon,R}^2(s)) \right) \mathcal{H}^1(\{d(x) = s\}) ds + \varepsilon C_A. \end{aligned} \quad (1.60)$$

Inserting the estimate for the one-dimensional profile from Lemma 1.4.2, we obtain

$$\begin{aligned} & \int_{-\eta_{\varepsilon,R}}^{\eta_{\varepsilon,R}} \left(\frac{\varepsilon |\xi'_{\varepsilon,R}(s)|^2}{2(1 - \xi_{\varepsilon,R}^2(s))} + \frac{1}{2\varepsilon} (1 - \xi_{\varepsilon,R}^2(s)) \right) \mathcal{H}^1(\{d(x) = s\}) ds \\ & \stackrel{(1.51)}{\leq} \sup_{-\eta_{\varepsilon,R} \leq s \leq \eta_{\varepsilon,R}} \mathcal{H}^1(\{d(x) = s\}) \left(2 + O\left(\frac{\varepsilon}{R}\right) \right). \end{aligned} \quad (1.61)$$

Since ∂A and the signed distance function d are smooth in $(\partial A)_R$, we have

$$\lim_{s \rightarrow 0} \mathcal{H}^1(\{d(x) = s\}) = \mathcal{H}^1(\partial A). \quad (1.62)$$

In the limit $\varepsilon \rightarrow 0$, then $R \rightarrow 0$, estimates (1.60), (1.61) and (1.53) hence imply

$$\limsup_{R \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2} |\nabla m_\varepsilon|^2 + \frac{1}{2\varepsilon} (1 - m_{\varepsilon,1}^2) \right) \leq 2\mathcal{H}^1(\partial A). \quad (1.63)$$

We now turn to the estimate of the nonlocal term in the energy F . As for the local terms, our strategy is to use the one-dimensional estimates from Lemma 1.4.2. Invoking the coarea formula twice and inserting (1.58), we get

$$\begin{aligned} & \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \frac{|m_{\varepsilon,1}(x) - m_{\varepsilon,1}(y)|^2}{|x - y|^3} dx dy \\ & \geq \int_{-R}^R \int_{\{x: d(x)=\rho'\}} \int_{-R}^R \int_{\{y: d(y)=\rho\}} \frac{|\xi_{\varepsilon,R}(\rho') - \xi_{\varepsilon,R}(\rho)|^2}{|x - y|^3} d\mathcal{H}^1(y) d\rho d\mathcal{H}^1(x) d\rho'. \end{aligned} \quad (1.64)$$

We claim that the integrals over curves tangential to the boundary may be estimated as follows: For every $\delta > 0$, there is an $R_{\delta,A}$ such that

$$\int_{\{x: d(x)=\rho'\}} \int_{\{y: d(y)=\rho\}} \frac{1}{|x - y|^3} d\mathcal{H}^1(y) d\mathcal{H}^1(x) \geq (1 - \delta) \frac{2\mathcal{H}^1(\partial A)}{(\rho - \rho')^2}, \quad (1.65)$$

for all $R \leq R_{\delta,A}$ and all $\rho \neq \rho' \in (-R, R)$. Assuming for a moment that (1.65) holds, we conclude by inserting (1.65) into (1.64) and applying the one-dimensional estimate (1.52)

$$\begin{aligned} & \frac{\lambda}{|\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla^{\frac{1}{2}} m_{\varepsilon,1}|^2 dx \\ & \stackrel{(1.64), (1.65)}{\geq} (1 - \delta) \frac{\lambda \mathcal{H}^1(\partial A)}{2\pi |\log \varepsilon|} \int_{-R}^R \int_{-R}^R \frac{|\xi_{\varepsilon,R}(\rho) - \xi_{\varepsilon,R}(\rho')|^2}{|\rho - \rho'|^2} d\rho' d\rho \\ & \stackrel{(1.52)}{\geq} (1 - \delta) 2\mathcal{H}^1(\partial A) \frac{\lambda \log(cR/\varepsilon)}{\lambda_c |\log \varepsilon|}. \end{aligned}$$

Since δ was arbitrary, we obtain

$$\liminf_{R \rightarrow 0} \liminf_{\varepsilon \rightarrow 0} \frac{\lambda}{|\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla^{\frac{1}{2}} m_{\varepsilon,1}|^2 dx \geq 2\mathcal{H}^1(\partial A) \frac{\lambda}{\lambda_c}. \quad (1.66)$$

Together, (1.63) and (1.66) imply the limsup inequality by a standard diagonal argument.

It remains to prove (1.65), for which we fix $x \in (\partial A)_R$ with $d(x) = \rho'$ and pass to curvilinear coordinates in a neighborhood of $\tilde{x} := p(x) \in \partial A$. More

precisely, let the curve $\gamma : (-R^{\frac{1}{2}}, R^{\frac{1}{2}}) \rightarrow \partial A$ be a parametrization by arclength of a neighborhood of \tilde{x} in ∂A with $\gamma(0) = \tilde{x}$. Then, for all $R \leq R_A$ with some $R_A > 0$ the function

$$\Psi(\sigma, \rho) := \gamma(\sigma) + \nu(\gamma(\sigma))\rho$$

is a diffeomorphism from $(-R^{\frac{1}{2}}, R^{\frac{1}{2}}) \times (-R, R)$ onto its image, which we denote by $\Gamma_{\tilde{x}}$. The choice $R^{\frac{1}{2}}$ will become clear later. Note that due to compactness of ∂A , we may choose R_A independent of \tilde{x} . A transformation of variables then yields

$$\int_{\{y: d(y)=\rho\} \cap \Gamma_{p(x)}} \frac{1}{|x-y|^3} d\mathcal{H}^1(y) = \int_{-R^{\frac{1}{2}}}^{R^{\frac{1}{2}}} \frac{(1 + \kappa(\gamma(\sigma))\rho)}{|\Psi(0, \rho') - \Psi(\sigma, \rho)|^3} d\sigma, \quad (1.67)$$

where $\kappa(\tilde{y})$ denotes the signed curvature of ∂A at \tilde{y} (negative if A is convex). Since the curvature of ∂A is bounded, there is, for any $\delta > 0$, an $R_{\delta, A} > 0$ such that for all $R \leq R_{\delta, A}$ we have

$$|\kappa|R \leq \delta \quad \text{and} \quad |\Psi(0, \rho') - \Psi(\sigma, \rho)| \leq (1 + \delta)\sqrt{\sigma^2 + (\rho - \rho')^2}. \quad (1.68)$$

We conclude that, for any $\tilde{\delta} > 0$, there is an $\tilde{R}_{\tilde{\delta}, A} > 0$ such that for all $R \leq \tilde{R}_{\tilde{\delta}, A}$ and all $\rho, \rho' \in (-R, R)$ we have

$$\begin{aligned} \int_{\{y: d(y)=\rho\} \cap \Gamma_{p(x)}} \frac{1}{|x-y|^3} d\mathcal{H}^1(y) \\ \stackrel{(1.67), (1.68)}{\geq} (1 - \tilde{\delta}) \int_{-R^{\frac{1}{2}}}^{R^{\frac{1}{2}}} \frac{1}{(\sigma^2 + (\rho - \rho')^2)^{3/2}} d\sigma \\ = (1 - \tilde{\delta}) \frac{2}{(\rho - \rho')^2} \frac{R^{\frac{1}{2}}}{\sqrt{R + (\rho - \rho')^2}} \geq (1 - 2\tilde{\delta}) \frac{2}{(\rho - \rho')^2}. \end{aligned} \quad (1.69)$$

Integrating (1.69) over x and invoking (1.62) we obtain (1.65). \square

1.4.2 Proof of Theorem 1.2.6

We begin with the proof of the lower bound in Theorem 1.2.6, which is the subject of Lemma 1.4.4. The proof of Theorem 1.2.6 is completed with the construction of the upper bound, carried out in Lemma 1.4.5.

Lemma 1.4.4. *Let $\lambda_c := \frac{\pi}{2}$ and $F_{\varepsilon, \lambda}$ as defined in (1.11). Then there is a universal constant $\delta > 0$ such that for all $\varepsilon < \frac{1}{2}$ and all*

$$\lambda_c \leq \lambda < \delta |\log \varepsilon| \quad (1.70)$$

the family of functionals $\{F_{\varepsilon, \lambda}\}$ is bounded below by

$$\min F_{\varepsilon, \lambda} \gtrsim -\frac{\lambda \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}}}{|\log \varepsilon|}. \quad (1.71)$$

Moreover, the profiles achieving the optimal scaling can be characterized as follows: For any $\gamma > 0$ and all $m \in H^1(\mathbb{T}^2; \mathbb{S}^2)$ which satisfy

$$F_{\varepsilon, \lambda}(m) \leq -\frac{\lambda \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}}}{|\log \varepsilon|} \gamma, \quad (1.72)$$

there holds

$$\int_{\mathbb{T}^2} |\nabla m_1| \, dx \leq \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2} |\nabla m|^2 + \frac{1 - m_{\varepsilon, 1}^2}{2\varepsilon} \right) dx \leq \frac{\lambda}{|\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla^{\frac{1}{2}} m_1|^2 \, dx, \quad (1.73)$$

and the above quantities agree to leading order and scale like $\varepsilon^{\frac{\lambda_c - \lambda}{\lambda}}$, i.e. if A and B are any of the three quantities in (1.73), we have

$$A \sim \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}} \quad \text{and} \quad |A - B| \lesssim \frac{\lambda}{|\log \varepsilon|} A, \quad (1.74)$$

where the constants may depend on γ .

Proof. By (1.29), we may bound the energy from below by

$$\begin{aligned} F_{\varepsilon, \lambda}(m) &\stackrel{(1.29)}{\geq} \left(1 - \frac{\lambda}{|\log \varepsilon|} \right) \int_{\mathbb{T}^2} \frac{\varepsilon}{2} |\nabla m|^2 + \frac{1}{2\varepsilon} (1 - m_1^2) \, dx \\ &\quad - \frac{\lambda \log \left(c_* \max \left\{ 1, \min \left\{ \frac{1}{\varepsilon \int_{\mathbb{T}^2} |\nabla m_1| \, dx}, \frac{1}{\varepsilon} \right\} \right\} \right)}{\lambda_c |\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla m_1| \, dx. \end{aligned} \quad (1.75)$$

Without loss of generality, we may assume that $\int_{\mathbb{T}^2} |\nabla m_1| \, dx > 0$. We first consider the case $\min \left\{ \frac{1}{\varepsilon \int_{\mathbb{T}^2} |\nabla m_1| \, dx}, \frac{1}{\varepsilon} \right\} \leq 1$, for which, with the help of (A.7), the estimate in (1.75) turns into

$$\begin{aligned} F_{\varepsilon, \lambda}(m) &\geq \left(1 - \frac{\lambda \log(c_*^{1/\lambda_c})}{|\log \varepsilon|} \right) \int_{\mathbb{T}^2} |\nabla m_1| \, dx \\ &\stackrel{(1.70)}{\geq} (1 - C\delta) \int_{\mathbb{T}^2} |\nabla m_1| \, dx \end{aligned} \quad (1.76)$$

for some universal constant $C > 0$. For $\delta < 1/C$, the right hand side of (1.76) is positive and the lower bound follows. Hence, we may assume

$\min\{\frac{1}{\varepsilon \int_{\mathbb{T}^2} |\nabla m_1| dx}, \frac{1}{\varepsilon}\} > 1$ so that (1.75) implies

$$\begin{aligned} F_{\varepsilon, \lambda}(m) &\geq \left(1 - \frac{\lambda}{|\log \varepsilon|}\right) \int_{\mathbb{T}^2} \frac{\varepsilon}{2} |\nabla m|^2 + \frac{1}{2\varepsilon} (1 - m_1^2) dx \\ &\quad - \frac{\lambda}{\lambda_c} \frac{\log\left(\frac{c_*}{\varepsilon \int_{\mathbb{T}^2} |\nabla m_1| dx}\right)}{|\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla m_1| dx. \end{aligned} \quad (1.77)$$

Abbreviating the energetic cost for m to deviate from the optimal Bloch wall profile by

$$D_\varepsilon(m) := \int_{\mathbb{T}^2} \frac{\varepsilon}{2} |\nabla m|^2 + \frac{1}{2\varepsilon} (1 - m_1^2) dx - \int_{\mathbb{T}^2} |\nabla m_1| dx,$$

and inserting $\mu := \varepsilon^{\frac{\lambda - \lambda_c}{\lambda}} \int_{\mathbb{T}^2} |\nabla m_1| dx$ and $c_{**} := c_* e^{\lambda_c}$ into the lower bound in (1.77), we get

$$F_{\varepsilon, \lambda}(m) \geq \left(1 - \frac{\lambda}{|\log \varepsilon|}\right) D_\varepsilon(m) - \frac{\lambda}{\lambda_c} \frac{\log\left(\frac{c_{**}}{\mu}\right)}{|\log \varepsilon|} \mu \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}}. \quad (1.78)$$

Since $\sup_{\mu > 0} \mu \log(c_{**}/\mu) = c_{**}/e$, and since $D_\varepsilon(m) \geq 0$ by (A.7), the lower bound in (1.71) follows.

We now turn to the proof of (1.74). Note that (A.7) and $F_{\varepsilon, \lambda}(m) \leq 0$ yield

$$\int_{\mathbb{T}^2} |\nabla m_1| dx \leq \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2} |\nabla m|^2 + \frac{1 - m_1^2}{2\varepsilon} \right) dx \leq \frac{\lambda}{|\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla^{\frac{1}{2}} m_1| dx.$$

For (1.74) it is hence sufficient to show

$$\begin{aligned} \int_{\mathbb{T}^2} |\nabla m_1| dx &\sim \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}}, \\ \frac{\lambda}{|\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla^{\frac{1}{2}} m_1| dx - \int_{\mathbb{T}^2} |\nabla m_1| dx &\lesssim \frac{\lambda \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}}}{|\log \varepsilon|}, \end{aligned} \quad (1.79)$$

where here and in the rest of the proof the constants may depend on γ . We combine the lower bound for the energy (1.78) with the upper bound (1.72) to obtain $\mu \log(c_{**}/\mu) \gtrsim 1$, which in turn implies $\mu \sim 1$. Hence, the first item in (1.79) may be estimated as

$$\int_{\mathbb{T}^2} |\nabla m_1| dx = \mu \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}} \sim \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}}.$$

For $\delta > 0$ sufficiently small universal and $\mu \sim 1$, the second item in (1.79) follows from (1.78):

$$\frac{\lambda}{|\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla^{\frac{1}{2}} m_1| dx - \int_{\mathbb{T}^2} |\nabla m_1| dx = -F_{\varepsilon, \lambda}(m) + D_{\varepsilon}(m) \stackrel{(1.78)}{\lesssim} \frac{\lambda \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}}}{|\log \varepsilon|}.$$

This concludes the proof. \square

Lemma 1.4.5 (Upper bound in the supercritical regime). *There is a constant $0 < K < 1$ such that for every (ε, λ) with*

$$\lambda_c < \lambda \quad \text{and} \quad 0 < \varepsilon^{\frac{\lambda - \lambda_c}{\lambda}} < K, \quad (1.80)$$

there is $m_{\varepsilon, \lambda} \in H^1(\mathbb{T}^2; \mathbb{S}^2)$ which satisfies

$$F_{\varepsilon, \lambda}(m_{\varepsilon, \lambda}) \lesssim -\frac{\lambda \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}}}{|\log \varepsilon|}.$$

Proof. We make an ansatz with N transitions equally separated by $1/N$ -sized regions of approximately constant magnetization. More precisely, we take the transitions as solutions of the optimal profile ODE and define

$$m_{\varepsilon, N}(x_1, x_2) = \begin{cases} \xi_{\varepsilon, \infty} \left(\frac{x_2 - \frac{1}{2N}}{\varepsilon} \right) e_1 + \sqrt{1 - \xi_{\varepsilon, \infty}^2 \left(\frac{x_2 - \frac{1}{2N}}{\varepsilon} \right)} e_2, & x_2 \in [0, \frac{1}{N}] \\ \xi_{\varepsilon, \infty} \left(\frac{\frac{3}{2N} - x_2}{\varepsilon} \right) e_1 + \sqrt{1 - \xi_{\varepsilon, \infty}^2 \left(\frac{\frac{3}{2N} - x_2}{\varepsilon} \right)} e_2, & x_2 \in [\frac{1}{N}, \frac{2}{N}] \end{cases}$$

extended periodically to \mathbb{T}^2 (see Fig. 1.2). Applying Lemma 1.4.2 with $X = \frac{1}{2N}$ and using symmetries of $m_{\varepsilon, N}$, we get

$$\int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2} |\nabla m_{\varepsilon, N}|^2 + \frac{1 - m_{(\varepsilon, N), 1}^2}{2\varepsilon} \right) dx \leq 2N \quad (1.81)$$

and, for all $\varepsilon < \frac{1}{4N}$, we have

$$\begin{aligned} & \int_{\mathbb{T}^2} |\nabla^{\frac{1}{2}} m_{(\varepsilon, N), 1}|^2 dx \\ & \stackrel{(A.11)}{=} \frac{1}{4\pi} \int_{\mathbb{T}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|m_{(\varepsilon, N), 1}(x_2) - m_{(\varepsilon, N), 1}(y_2)|^2}{(|x_2 - y_2|^2 + s^2)^{3/2}} ds dx_2 dy_2 \\ & \geq \frac{1}{2\pi} \sum_{k=1}^N \int_{\frac{k-1}{N}}^{\frac{k}{N}} \int_{\frac{k-1}{N}}^{\frac{k}{N}} \frac{|m_{(\varepsilon, N), 1}(x_2) - m_{(\varepsilon, N), 1}(y_2)|^2}{|x_2 - y_2|^2} dx_2 dy_2 \\ & = \frac{N}{4\lambda_c} \int_{-\frac{1}{2N}}^{\frac{1}{2N}} \int_{-\frac{1}{2N}}^{\frac{1}{2N}} \frac{|\xi_{\varepsilon, \infty}(x) - \xi_{\varepsilon, \infty}(y)|^2}{|x - y|^2} dx dy \stackrel{(1.52)}{\geq} 2N \frac{\log(\frac{c}{2\varepsilon N})}{\lambda_c}. \end{aligned} \quad (1.82)$$

To obtain the upper bound, we combine estimates (1.81) and (1.82) and optimize in $N \in \mathbb{N}$. The choice $N := 2 \left\lfloor K \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}} \right\rfloor$ is admissible because $N \stackrel{(1.80)}{\geq} 2$ and $\varepsilon N \leq 2K \leq \frac{1}{4}$ for $K \leq \frac{1}{8} \min\{1, c\}$. Since $0 < \varepsilon < 1$, we get

$$F_{\varepsilon, \lambda}(m_{\varepsilon, N}) \leq 2N \left(1 - \frac{\lambda \log(\frac{c}{2\varepsilon N})}{\lambda_c |\log \varepsilon|} \right) \leq - \frac{C \lambda \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}}}{|\log \varepsilon|}, \quad (1.83)$$

for some universal $C > 0$, which is the desired estimate. \square

1.4.3 Proof of Theorem 1.2.7

Theorem 1.2.7. We start by proving item (i). Inserting (1.27) into the lower bound (1.45), we get for sufficiently small $\varepsilon > 0$

$$\begin{aligned} F_{\varepsilon, \lambda}(m) &\geq \left(1 - \frac{\log(\varepsilon c) \log(\varepsilon/\beta_1)}{\log(\varepsilon)^2} \right) \int_{\mathbb{T}^2} |\nabla m_1| \, dx \\ &\geq \left(\frac{|\log(\varepsilon)| \log(c/\beta_1) + \log(c) \log(\beta_1)}{|\log(\varepsilon)|^2} \right) \int_{\mathbb{T}^2} |\nabla m_1| \, dx. \end{aligned}$$

For $\beta_1 < c$, the bracket is positive, which shows that the minimal value of $\min F_{\varepsilon, \lambda} = 0$ is only attained for $m \equiv \pm e_1$. Since $\varepsilon^{\frac{\lambda_+(\varepsilon) - \lambda_c}{\lambda_+(\varepsilon)}} \leq \frac{2}{\beta_2}$ for sufficiently small $\varepsilon > 0$, the second part follows from Lemma 1.4.5.

To proceed, we next establish the estimate

$$\int_{\mathbb{T}^2} |\nabla m_1| \, dx \lesssim \max\{1, |\log \varepsilon| F_{\varepsilon, \lambda_c}(m)\}. \quad (1.84)$$

It is enough to show that there are constants $C, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have

$$\int_{\mathbb{T}^2} |\nabla m_1| \, dx \geq C \quad \implies \quad F_{\varepsilon, \lambda_c}(m) \gtrsim \frac{1}{|\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla m_1| \, dx. \quad (1.85)$$

Indeed, by (1.29), we may bound the energy from below by

$$\begin{aligned} F_{\varepsilon, \lambda_c}(m_\varepsilon) &\stackrel{(1.29)}{\geq} \left(1 - \frac{\lambda_c}{|\log \varepsilon|} \right) \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2} |\nabla m_\varepsilon|^2 + \frac{1}{2\varepsilon} (1 - m_{\varepsilon, 1}^2) \right) dx \\ &\quad - \frac{\log \left(c_* \max \left\{ 1, \min \left\{ \frac{1}{\varepsilon \int_{\mathbb{T}^2} |\nabla m_{\varepsilon, 1}| \, dx}, \frac{1}{\varepsilon} \right\} \right\} \right)}{|\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla m_{\varepsilon, 1}| \, dx. \end{aligned} \quad (1.86)$$

We first consider the case $\min\{\frac{1}{\varepsilon \int_{\mathbb{T}^2} |\nabla m_{\varepsilon, 1}| \, dx}, \frac{1}{\varepsilon}\} \leq 1$, for which (1.86) turns into

$$F_{\varepsilon, \lambda_c}(m) \geq \left(1 - \frac{\lambda_c + \log(c_*)}{|\log \varepsilon|} \right) \int_{\mathbb{T}^2} |\nabla m_{\varepsilon, 1}| \, dx \gtrsim \int_{\mathbb{T}^2} |\nabla m_1| \, dx.$$

For the remaining case, we have $\min\{\frac{1}{\varepsilon \int_{\mathbb{T}^2} |\nabla m_1| dx}, \frac{1}{\varepsilon}\} \geq 1$ and (1.86) implies

$$\begin{aligned} F_{\varepsilon, \lambda_c}(m_\varepsilon) &\geq \left(1 - \frac{\lambda_c}{|\log \varepsilon|}\right) \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2} |\nabla m_\varepsilon|^2 + \frac{1}{2\varepsilon} (1 - m_{\varepsilon,1}^2)\right) dx \\ &\quad - \frac{\log\left(\frac{c_*}{\varepsilon \int_{\mathbb{T}^2} |\nabla m_{\varepsilon,1}| dx}\right)}{|\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla m_{\varepsilon,1}| dx \\ &\stackrel{(A.7)}{\geq} - \frac{\log\left(\frac{c_{**}}{\int_{\mathbb{T}^2} |\nabla m_1| dx}\right)}{|\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla m_1| dx, \end{aligned}$$

where we have inserted $c_{**} := c_* e^{\lambda_c}$. The estimate (1.85) follows with the choice $C = 2c_{**}$.

With (1.84) at hand, we now prove item (ii), starting with the lower bound. Let $m_\varepsilon \rightarrow m$ in $L^1(\mathbb{T}^2)$ for some $m \in L^1(\mathbb{T}^2; \mathbb{R}^3)$. Lemma 1.4.4 yields

$$\liminf_{\varepsilon \rightarrow 0} F_{\varepsilon, \lambda_c}(m_\varepsilon) \geq 0,$$

which proves the lower bound in case that $m \in L^1(\mathbb{T}^2; \{\pm e_1\})$. For the remaining case, we may assume $\int_{\mathbb{T}^2} (1 - m_{\varepsilon,1}^2) dx \gtrsim 1$. For sufficiently small ε , estimates (1.29) and (1.84) then yield

$$\begin{aligned} \int_{\mathbb{T}^2} (1 - m_{\varepsilon,1}^2) dx &\lesssim \varepsilon \left(F_{\varepsilon, \lambda}(m_\varepsilon) + \frac{\lambda_c}{|\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla^{\frac{1}{2}} m_{\varepsilon,1}|^2 dx \right) \\ &\stackrel{(1.29)}{\lesssim} \varepsilon \left(F_{\varepsilon, \lambda}(m_\varepsilon) + \int_{\mathbb{T}^2} |\nabla m_{\varepsilon,1}| dx \right) \stackrel{(1.84)}{\lesssim} \varepsilon (1 + |\log \varepsilon| F_{\varepsilon, \lambda_c}(m_\varepsilon)), \end{aligned} \tag{1.87}$$

which implies $\liminf_{\varepsilon \rightarrow 0} F_{\varepsilon, \lambda_c}(m_\varepsilon) = +\infty$ for $m \in L^1(\mathbb{T}^2; \mathbb{R}^3) \setminus L^1(\mathbb{T}^2; \{\pm e_1\})$. Since the construction of the upper bound was already carried out in Lemma 1.4.3, the proof is complete.

To prove item (iii), we again make use of the construction in Lemma 1.4.5. However, this time we take $N = \lfloor \log(|\log \varepsilon|) \rfloor$. Analogous to (1.83), we get for sufficiently small ε

$$F_{\varepsilon, \lambda}(m_{\varepsilon, N}) \leq 2N \left(1 - \frac{\log(\frac{2\varepsilon N}{c})}{\log \varepsilon}\right) \lesssim \frac{N \log N}{|\log \varepsilon|} \rightarrow 0, \quad \text{for } \varepsilon \rightarrow 0.$$

Therefore, it remains to show that $m_{\varepsilon, N}$ is not compact in the strong L^1 -topology. Since $\int_{\mathbb{T}^2} |m_{\varepsilon, N}|^2 dx = 1$, any possible limit \tilde{m} of (a subsequence of) $m_{\varepsilon, N}$ in the strong topology needs to satisfy $\int_{\mathbb{T}^2} |\tilde{m}|^2 dx = 1$. However, since $\varepsilon N \rightarrow 0$ as $\varepsilon \rightarrow 0$, it is clear that $m_{\varepsilon, N}$ converges weakly to zero in $L^2(\mathbb{T}^2)$, leading to a contradiction. Finally, item (iv) follows directly from (1.87), (1.84) and the compact embedding $BV(\mathbb{T}^2) \hookrightarrow L^1(\mathbb{T}^2)$. \square

1.5 Stray field estimates and reduction of the full energy

The goal of this section is to make the heuristic reduction in section 1.1.1 rigorous. We prove the following

Lemma 1.5.1 (Reduction of the energy). *There is a universal constant $C > 0$ such that energy E is bounded below by*

$$\begin{aligned} E(m) \geq \ell^2 t + (1 - Ct^2) \int_{(0,t) \times \mathbb{T}_\ell^2} |\nabla m|^2 + (Q - 1)(m_2^2 + m_3^2) \, dx \\ - 2 \int_{(0,t) \times \mathbb{T}_\ell^2} m_1 h_{\text{ext}} \, dx - \frac{t^2}{2} \int_{\mathbb{T}_\ell^2} |\nabla^{\frac{1}{2}} \bar{m}_1|^2 \, dx, \end{aligned} \quad (1.88)$$

where $\bar{m}(x') = \frac{1}{t} \int_0^t m(x_1, x') \, dx_1$ denotes the e_1 -average of the magnetization over $(0, t)$.

Note that for two-dimensional magnetizations (1.88) also holds in the reversed direction if $-C$ is replaced by C . Hence the lower bound is asymptotically sharp. We also remark that a similar sharp estimate for the three-dimensional dipolar energy holds for thin three-dimensional domains in the whole space [75].

For the proof of Lemma 1.5.1, which is deferred until the end of this section, we need several estimates presented in the following sections.

1.5.1 Approximation of m by its e_1 -average \bar{m}

Since the thickness t of the film is small, the exchange energy strongly penalizes oscillations of the magnetization in the normal direction of the film. Hence the averaged magnetization \bar{m} is a good approximation of m , and Assumption (i) in section 1.1.1 can be made rigorous by the following Poincaré-type inequality

$$\int_{(0,t) \times \mathbb{T}_\ell^2} |m - \chi_{(0,t)} \bar{m}|^2 \, dx \lesssim t^2 \int_{(0,t) \times \mathbb{T}_\ell^2} |\partial_1 m|^2 \, dx, \quad (1.89)$$

which holds for all $m \in H^1((0, t) \times \mathbb{T}_\ell^2; \mathbb{R}^3)$ and can be proved by standard methods.

1.5.2 Approximation of the stray field energy

In this section, we establish an approximation of the stray field, i.e. a rigorous version of Assumption (i). In particular, we show that for thin films, the difference between the stray field energy of the averaged magnetization and the stray field energy of the full magnetization may be estimated by the exchange energy at lower order. The statement of Theorem 1.5.2 below is slightly stronger than what is necessary to prove Lemma 1.5.1 and might be of independent interest for other thin film regimes.

Theorem 1.5.2. *Let $m \in H^1((0, t) \times \mathbb{T}_\ell^2; \mathbb{R}^3)$, then the stray field energy (see (1.3)) satisfies*

$$\left| \int_{\mathbb{R} \times \mathbb{T}_\ell^2} |h[m]|^2 dx - \int_{\mathbb{R} \times \mathbb{T}_\ell^2} |h[m_1 e_1]|^2 dx - \int_{\mathbb{R} \times \mathbb{T}_\ell^2} |h[m']|^2 dx \right| \lesssim t^2 \int_{(0, t) \times \mathbb{T}_\ell^2} |\nabla m|^2 dx, \quad (1.90)$$

$$\left| \int_{\mathbb{R} \times \mathbb{T}_\ell^2} |h[m]|^2 dx - \int_{\mathbb{R} \times \mathbb{T}_\ell^2} |h[\chi_{(0, t)} \overline{m}]|^2 dx \right| \lesssim t^2 \int_{(0, t) \times \mathbb{T}_\ell^2} |\nabla m|^2 dx, \quad (1.91)$$

where $m' = m - m_1 e_1$ is understood to have values in \mathbb{R}^3 with e_1 -component 0. Moreover, the contributions due to m_1 and m' may be approximated by

$$\left| \int_{\mathbb{R} \times \mathbb{T}_\ell^2} |h[m_1 e_1]|^2 dx - \int_{(0, t) \times \mathbb{T}_\ell^2} m_1^2 dx + \frac{t^2}{2} \int_{\mathbb{T}_\ell^2} |\nabla^{\frac{1}{2}} \overline{m}_1|^2 dx \right| \lesssim t^2 \int_{(0, t) \times \mathbb{T}_\ell^2} |\nabla m|^2 dx, \quad (1.92)$$

$$\left| \int_{\mathbb{R} \times \mathbb{T}_\ell^2} |h[m']|^2 dx - \frac{t^2}{2} \int_{\mathbb{T}_\ell^2} |\nabla^{-\frac{1}{2}} \nabla' \cdot \overline{m}'|^2 dx \right| \lesssim t^2 \int_{(0, t) \times \mathbb{T}_\ell^2} |\nabla m|^2 dx, \quad (1.93)$$

$$\int_{\mathbb{R} \times \mathbb{T}_\ell^2} |h[m']|^2 dx \lesssim t^2 \int_{(0, t) \times \mathbb{T}_\ell^2} (|\nabla m|^2 + |m'|^2) dx. \quad (1.94)$$

Proof. It is sufficient to argue for $m \in C_c^\infty(\mathbb{R} \times \mathbb{T}_\ell^2; \mathbb{R}^3)$, because the general case follows by an approximation argument, as we now explain. Since $(0, t) \times \mathbb{T}_\ell^2$ is an extension domain, there exists, for every $m \in H^1((0, t) \times \mathbb{T}_\ell^2; \mathbb{R}^3)$, a sequence $\{m_k\}_{k \in \mathbb{N}}$ in $C_c^\infty(\mathbb{R} \times \mathbb{T}_\ell^2; \mathbb{R}^3)$ such that $\|m - m_k\|_{L^2(\mathbb{R} \times \mathbb{T}_\ell^2)} + \|\nabla m -$

$\nabla m_k \|_{L^2((0,t) \times \mathbb{T}_\ell^2)} \rightarrow 0$. It remains to check that all terms in (1.90) – (1.93) are continuous. Note that by (1.89), we also have $\|\bar{m}_k - \bar{m}\|_{L^2(\mathbb{T}_\ell^2)} \rightarrow 0$. Moreover, $t \int_{\mathbb{T}_\ell} |\nabla \bar{m}_k|^2 dx \lesssim \int_{(0,t) \times \mathbb{T}_\ell^2} |\nabla m_k|^2 dx$ (see (A.8) in the Appendix for a proof). Hence the convergence follows from the elliptic estimate $\int_{\mathbb{R} \times \mathbb{T}_\ell^2} |h[m_k - m]|^2 dx \leq \int_{\mathbb{R} \times \mathbb{T}_\ell^2} |m_k - m|^2 dx$ and by interpolation for the terms involving fractional derivatives.

We write the stray field energy in terms of the magnetostatic potential φ

$$\int_{\mathbb{R} \times \mathbb{T}_\ell^2} |h[m]|^2 dx = - \int_{\mathbb{R} \times \mathbb{T}_\ell^2} \varphi \nabla \cdot m dx \quad \text{where } \Delta \varphi = \nabla \cdot m \text{ in } \mathcal{D}'(\mathbb{R} \times \mathbb{T}_\ell^2).$$

Upon passing to Fourier series (with respect to the in-plane variables), we get

$$\int_{\mathbb{R} \times \mathbb{T}_\ell^2} \varphi \nabla \cdot m dx = \frac{1}{\ell^2} \int_{\mathbb{R}} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} \hat{\varphi}_k^*(z) (\partial_z \hat{m}_{1,k}(z) - ik \cdot \hat{m}'_k(z)) dz, \quad (1.95)$$

where the Fourier coefficients $\hat{\varphi}_k$ of φ with $\hat{\varphi}_k : \mathbb{R} \rightarrow \mathbb{C}$ for $k \in \frac{2\pi}{\ell} \mathbb{Z}^2$ solve

$$\partial_z^2 \hat{\varphi}_k - |k|^2 \hat{\varphi}_k = \partial_z \hat{m}_{1,k} - ik \cdot \hat{m}'_k.$$

We introduce the fundamental solution

$$H_k(s) = \begin{cases} \frac{e^{-|k||s|}}{|k|} & \text{for } k \neq 0, \\ -|s| & \text{for } k = 0, \end{cases}$$

which satisfies

$$-\partial_s^2 H_k + |k|^2 H_k = 2\delta \quad \text{in } \mathcal{D}'(\mathbb{R}) \text{ for all } k \in \mathbb{Z}^2, \quad (1.96)$$

where δ denotes the Dirac measure at 0. The fundamental solution allows to rewrite $\hat{\varphi}_k(z)$ as

$$\hat{\varphi}_k(z) = -\frac{1}{2} \int_{\mathbb{R}} H_k(z - z') (\partial_z \hat{m}_{1,k}(z') - ik \cdot \hat{m}'_k(k, z')) dz',$$

which by (1.95) leads to the following expression for the stray field energy

$$\begin{aligned} \int_{\mathbb{R} \times \mathbb{T}_\ell^2} |h[m]|^2 dx &= \frac{1}{2\ell^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} (\partial_z \hat{m}_{1,k}(z) - ik \cdot \hat{m}'_k(z))^* \\ &\quad \times H_k(z - z') (\partial_z \hat{m}_{1,k}(z') - ik \cdot \hat{m}'_k(z')) dz dz'. \end{aligned} \quad (1.97)$$

To prove (1.90), we need to show that the mixed terms in (1.97), i.e. terms of the form

$$I := \frac{1}{\ell^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} \partial_z \widehat{m}_{1,k}^*(z) H_k(z - z') (ik \cdot \widehat{m}'_k(z')) dz dz' \quad (1.98)$$

satisfy $|I| \lesssim t^2 \int_{(0,t) \times \mathbb{T}_{\ell}^2} |\nabla m|^2 dx$. Integrating by parts in (1.98), we get

$$I = -\frac{1}{\ell^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} \widehat{m}_{1,k}^*(z) \partial_z H_k(z - z') (ik \cdot \widehat{m}'_k(z')) dz dz'. \quad (1.99)$$

We write $m = \chi_{(0,t)} \overline{m} + u$ where as usual $\overline{m}(x') = \frac{1}{t} \int_0^t m(x_1, x') dx_1$ denotes the average of m over in the e_1 -direction. With this notation, (1.99) turns into

$$\begin{aligned} I = -\frac{1}{\ell^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} (\chi_{(0,t)}(z) \widehat{\overline{m}}_{1,k} + \widehat{u}_{1,k}(z))^* \partial_z H(k, z - z') \\ \times \left(ik \cdot \chi_{(0,t)}(z') \widehat{\overline{m}}'_k + ik \cdot \widehat{u}'_k(z') \right) dz dz'. \end{aligned} \quad (1.100)$$

Since $\partial_s H_k(s) = -\frac{s}{|s|} e^{-|k||s|}$ is anti-symmetric in s , we have $\int_0^t \int_0^t \partial_z H_k(z - z') dz dz' = 0$ which means that upon expanding (1.100), the term involving \overline{m}_1 and \overline{m}' vanishes. Furthermore, we have $|\partial_z H_k| \leq 1$ and hence the remaining terms in (1.100) may be estimated by

$$\begin{aligned} |I| \leq \frac{1}{\ell^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} \left(|\widehat{u}_{1,k}(z)| |k \cdot \widehat{m}'_k(z')| \right. \\ \left. + |\chi_{(0,t)}(z) \widehat{\overline{m}}_{1,k}| |k \cdot \widehat{u}'_k(z')| \right) dz dz'. \end{aligned} \quad (1.101)$$

Note that passing to Fourier series in the in-plane variables commutes with taking e_1 -averages. Thus $\widehat{u}_{j,k}$ has e_1 -average zero for all $j = 1, 2, 3$ and the intermediate value theorem yields $\tau_{j,k}, \rho_{j,k} \in (0, t)$ such that $\Re \widehat{u}_{j,k}(\tau_{j,k}) = 0$ and $\Im \widehat{u}_{j,k}(\rho_{j,k}) = 0$. By the fundamental theorem of calculus, we hence get the estimate

$$|\widehat{u}_{j,k}(z)| \lesssim \int_0^t |\partial_z \widehat{m}_{j,k}(\tau)| d\tau \quad \text{for all } z \in (0, t) \text{ and } j = 1, 2, 3. \quad (1.102)$$

Inserting (1.102) into (1.101) and using Jensen's inequality yields the rough estimate

$$|I| \lesssim \sum_{n,j=1}^3 \frac{t}{\ell^2} \int_0^t \int_0^t \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} |\partial_z \widehat{m}_{j,k}(z)| |k| |\widehat{m}_{n,k}(z')| dz dz'.$$

By Young's inequality and Parseval's identity, we conclude

$$\begin{aligned} |I| &\lesssim \sum_{n,j=1}^3 \frac{t}{\ell^2} \int_0^t \int_0^t \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} (|\partial_z \widehat{m}_{j,k}(z)|^2 + |k|^2 |\widehat{m}_{n,k}(z')|^2) \, dz \, dz' \\ &\lesssim t^2 \int_{(0,t) \times \mathbb{T}_\ell^2} |\nabla m|^2 \, dx, \end{aligned}$$

which completes the proof of (1.90). Assuming for a moment that (1.92) and (1.93) hold, identity (1.91) is obtained as follows. Applying (1.90) to m and $\chi_{(0,t)} \overline{m}$, we get

$$\begin{aligned} &\left| \int_{\mathbb{R} \times \mathbb{T}_\ell^2} |h[m]|^2 \, dx - \int_{\mathbb{R} \times \mathbb{T}_\ell^2} |h[\chi_{(0,t)} \overline{m}]|^2 \, dx \right. \\ &\quad - \int_{\mathbb{R} \times \mathbb{T}_\ell^2} |h[m_1 e_1]|^2 \, dx + \int_{\mathbb{R} \times \mathbb{T}_\ell^2} |h[\chi_{(0,t)} \overline{m}_1 e_1]|^2 \, dx \\ &\quad \left. - \int_{\mathbb{R} \times \mathbb{T}_\ell^2} |h[m']|^2 \, dx + \int_{\mathbb{R} \times \mathbb{T}_\ell^2} |h[\chi_{(0,t)} \overline{m}']|^2 \, dx \right| \\ &\quad \stackrel{(1.90)}{\lesssim} t^2 \int_{(0,t) \times T_\ell^2} |\nabla m|^2 \, dx, \end{aligned} \tag{1.103}$$

where we have also used (see (A.8) in the appendix for a proof)

$$\int_{(0,t) \times T_\ell^2} |\nabla (\chi_{(0,t)} \overline{m})|^2 \, dx = t \int_{\mathbb{T}_\ell^2} |\nabla' \overline{m}|^2 \, dx \stackrel{(A.8)}{\leq} \int_{(0,t) \times T_\ell^2} |\nabla m|^2 \, dx.$$

Applying (1.92) and (1.93) to (1.103) yields the claim

$$\begin{aligned} &\left| \int_{\mathbb{R} \times \mathbb{T}_\ell^2} |h[m]|^2 \, dx - \int_{\mathbb{R} \times \mathbb{T}_\ell^2} |h[\chi_{(0,t)} \overline{m}]|^2 \, dx \right. \\ &\quad \left. - \int_{(0,t) \times \mathbb{T}_\ell^2} m_1^2 \, dx + \int_{(0,t) \times \mathbb{T}_\ell^2} (\chi_{(0,t)} \overline{m}_1)^2 \, dx \right| \\ &\quad \stackrel{(1.92), (1.93)}{\lesssim} t^2 \int_{(0,t) \times \mathbb{T}_\ell^2} |\nabla m|^2 \, dx \\ &\quad \left| \int_{\mathbb{R} \times \mathbb{T}_\ell^2} |h[m]|^2 \, dx - \int_{\mathbb{R} \times \mathbb{T}_\ell^2} |h[\chi_{(0,t)} \overline{m}]|^2 \, dx \right| \stackrel{(1.89)}{\lesssim} t^2 \int_{(0,t) \times \mathbb{T}_\ell^2} |\nabla m|^2 \, dx. \end{aligned}$$

We turn to the proof of (1.92). Integrating by parts twice and inserting (1.96),

we get

$$\begin{aligned}
& \int_{\mathbb{R} \times \mathbb{T}_\ell^2} |h[m_1]|^2 \, dx \\
& \stackrel{(1.97)}{=} \frac{1}{2\ell^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} \partial_z \widehat{m}_{1,k}^*(z) H_k(z - z') \partial_z \widehat{m}_{1,k}(z') \, dz \, dz' \\
& = -\frac{1}{2\ell^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} \widehat{m}_{1,k}^*(z) \partial_z^2 H_k(z - z') \widehat{m}_{1,k}(z') \, dz \, dz' \\
& \stackrel{(1.96)}{=} \frac{1}{\ell^2} \int_{\mathbb{R}} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} |\widehat{m}_{1,k}(z)|^2 \, dz \\
& \quad - \frac{1}{2\ell^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} \widehat{m}_{1,k}^*(z) |k| e^{-|k||z-z'|} \widehat{m}_{1,k}(z') \, dz \, dz'.
\end{aligned}$$

Since $|1 - e^{-|k||z|}| \leq |k|t$ for $z \in (-t, t)$, the last line above

$$J := \frac{1}{2\ell^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} \widehat{m}_{1,k}^*(z) |k| e^{-|k||z-z'|} \widehat{m}_{1,k}(z') \, dz \, dz'$$

may be estimated, with the help of Young's inequality, by

$$\begin{aligned}
& \left| J - \frac{t^2}{2\ell^2} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} |k| |\widehat{m}_{1,k}(z)|^2 \right| \\
& \lesssim \frac{t}{\ell^2} \int_0^t \int_0^t \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} |\widehat{m}_{1,k}(z)| |k|^2 |\widehat{m}_{1,k}(z')| \, dz \, dz' \\
& \lesssim \frac{t^2}{\ell^2} \int_0^t \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} |k|^2 |\widehat{m}_{1,k}(z)|^2 \, dz,
\end{aligned}$$

which by Parseval's identity is equivalent to

$$\left| J - \frac{t^2}{2} \int_{\mathbb{T}_\ell^2} |\nabla^{\frac{1}{2}} \overline{m}_1|^2 \, dx \right| \lesssim t^2 \int_{(0,t) \times \mathbb{T}_\ell^2} |\nabla' m_1|^2 \, dx.$$

In total, we get

$$\begin{aligned}
& \left| \int_{\mathbb{R} \times \mathbb{T}_\ell^2} |h[m_1 e_1]|^2 \, dx - \int_{(0,t) \times \mathbb{T}_\ell^2} m_1^2 \, dx + \frac{t^2}{2} \int_{\mathbb{T}_\ell^2} |\nabla^{\frac{1}{2}} \overline{m}_1|^2 \, dx \right| \\
& \lesssim t^2 \int_{(0,t) \times \mathbb{T}_\ell^2} |\nabla' m_1|^2 \, dx,
\end{aligned}$$

which proves (1.92). We continue with the proof of (1.93). Since $|1 - e^{-|k||z|}| \leq |k|t$ for $z \in (0, t)$, we may insert $|H_k(z - z') - \frac{1}{|k|}| \leq t$ for $k \neq 0$ into (1.97)

$$\begin{aligned} & \int_{\mathbb{R} \times \mathbb{T}_\ell^2} |h[m']|^2 \, dx \\ & \stackrel{(1.97)}{=} \frac{1}{2\ell^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2 \setminus \{0\}} (k \cdot \widehat{m}'_k(z))^* H_k(z - z') k \cdot \widehat{m}'_k(z') \, dz \, dz'. \end{aligned}$$

This yields

$$\begin{aligned} & \left| \int_{\mathbb{R} \times \mathbb{T}_\ell^2} |h[m']|^2 \, dx - \frac{t}{2\ell^2} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2 \setminus \{0\}} \frac{|k \cdot \widehat{m}'_k|^2}{|k|} \right| \\ & \lesssim \frac{t^2}{2\ell^2} \int_{\mathbb{R}} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} |k \cdot \widehat{m}'_k(z)|^2 \, dz, \end{aligned}$$

which proves the first equality. The second equality follows as in (1.5). \square

Lemma 1.5.1. We invoke Theorem 1.5.2 to obtain a lower bound for the stray field energy. Combining (1.90) with (1.92) and neglecting the non-negative term $\int_{\mathbb{R} \times \mathbb{T}_\ell^2} |h[m']|^2 \, dx$, we get

$$\begin{aligned} \int_{\mathbb{R} \times \mathbb{T}_\ell^2} |h[m]|^2 \, dx & \stackrel{(1.90)}{\geq} \int_{\mathbb{R} \times \mathbb{T}_\ell^2} |h[m_1 e_1]|^2 \, dx - Ct^2 \int_{(0,t) \times \mathbb{T}_\ell^2} |\nabla m|^2 \, dx \\ & \stackrel{(1.92)}{\geq} \int_{(0,t) \times \mathbb{T}_\ell^2} m_1^2 \, dx - \frac{t^2}{2} \int_{\mathbb{T}_\ell^2} |\nabla^{\frac{1}{2}} \overline{m}_1|^2 \, dx \\ & \quad - Ct^2 \int_{(0,t) \times \mathbb{T}_\ell^2} |\nabla m|^2 \, dx, \end{aligned} \tag{1.104}$$

for some universal constant $C > 0$. Note that estimating $\int_{\mathbb{R} \times \mathbb{T}_\ell^2} |h[m']|^2 \, dx$ by zero is reasonable, since (1.93) shows that the term is controlled by the exchange and anisotropy energy at lower order. Inserting (1.104) into the energy E yields

$$\begin{aligned} E(m) & \stackrel{(1.2)}{=} \int_{(0,t) \times \mathbb{T}_\ell^2} (|\nabla m|^2 + Q(m_2^2 + m_3^2) - 2m_1 h_{\text{ext}}) \, dx + \int_{\mathbb{R} \times \mathbb{T}_\ell^2} |h|^2 \, dx. \\ & \stackrel{(1.104)}{\geq} \int_{(0,t) \times \mathbb{T}_\ell^2} (|\nabla m|^2 + Q(m_2^2 + m_3^2) - 2m_1 h_{\text{ext}}) \, dx + \int_{(0,t) \times \mathbb{T}_\ell^2} m_1^2 \, dx \\ & \quad - \frac{t^2}{2} \int_{\mathbb{T}_\ell^2} |\nabla^{\frac{1}{2}} \overline{m}_1|^2 \, dx - Ct^2 \int_{(0,t) \times \mathbb{T}_\ell^2} |\nabla m|^2 \, dx. \end{aligned} \tag{1.105}$$

The constraint $|m| = 1$ allows to combine the leading order of the stray field energy with the anisotropy energy which leads to constant contribution and a renormalized anisotropy term

$$\begin{aligned} \int_{(0,t) \times \mathbb{T}_\ell^2} Q(m_2^2 + m_3^2) dx + t \int_{\mathbb{T}_\ell^2} m_1^2 dx \\ = \ell^2 t + \int_{(0,t) \times \mathbb{T}_\ell^2} (Q - 1)(m_2^2 + m_3^2) dx. \end{aligned} \quad (1.106)$$

Finally, we insert (1.106) into (1.105) to extract the leading order constant $\ell^2 t$ and conclude the claim of Lemma 1.5.1

$$\begin{aligned} E(m) \geq \ell^2 t + \int_{(0,t) \times \mathbb{T}_\ell^2} (|\nabla m|^2 + (Q - 1)(m_2^2 + m_3^2) - 2m_1 h_{\text{ext}}) dx \\ - \frac{t^2}{2} \int_{\mathbb{T}_\ell^2} |\nabla^{\frac{1}{2}} \bar{m}_1|^2 dx - Ct^2 \int_{(0,t) \times \mathbb{T}_\ell^2} |\nabla m|^2 dx, \end{aligned}$$

which completes the proof. \square

1.6 Proofs for the full energy E

The proofs for the full energy E are based on the arguments in the proofs for the reduced energy F . We recommend to read section 1.4 first.

Under mild assumptions on ℓ, t, Q and h_{ext} , weaker than those of Theorems 1.2.1 – 1.2.4, Lemma 1.3.1 and Theorem 1.5.2 yield the following estimates for the rescaled energy J .

Lemma 1.6.1. *There are universal constants $C, \delta > 0$ such that for $(\ell, t, Q, h_{\text{ext}})$ which satisfy*

$$Q > 1, \quad t < \delta \min\{1, \ell\} \quad \text{and} \quad \frac{\ell}{\sqrt{Q-1}} h_{\text{ext}}(\ell x') = g(x') \quad (1.107)$$

for some $g \in L^1(\mathbb{T}^2)$, the rescaled energy J (see (1.13)) satisfies

$$\begin{aligned} J(m) \geq \left(1 - Ct^2 - \frac{t}{4\sqrt{Q-1}}\right) \int_{(0,1) \times \mathbb{T}^2} \left(\varepsilon |\nabla m|^2 + \frac{1}{\varepsilon} (m_2^2 + m_3^2)\right) dx \\ + \frac{1}{2\varepsilon t^2 (Q-1)} \int_{(0,1) \times \mathbb{T}^2} |\partial_1 m|^2 dx - 2 \int_{\mathbb{T}^2} g \bar{m}_1 dx \\ - \frac{t}{\pi \sqrt{Q-1}} \log \left(c_* \max \left\{ 1, \min \left\{ \frac{1}{\varepsilon \int_{\mathbb{T}^2} |\nabla m_1| dx}, \frac{1}{\varepsilon} \right\} \right\} \right) \int_{\mathbb{T}^2} |\nabla \bar{m}_1| dx, \end{aligned} \quad (1.108)$$

for all $m \in H^1((0,1) \times \mathbb{T}^2; \mathbb{S}^2)$, where we have abbreviated $\varepsilon := \frac{1}{\ell\sqrt{Q-1}}$. Furthermore, for any $\bar{m} \in H^1(\mathbb{T}^2; \mathbb{S}^2)$ we have the upper bound

$$\begin{aligned} J(\chi_{(0,1)}\bar{m}) &\leq (1 + Ct^2) \int_{\mathbb{T}^2} \left(\varepsilon |\nabla \bar{m}|^2 + \frac{1}{\varepsilon} (\bar{m}_2^2 + \bar{m}_3^2) \right) dx \\ &\quad - 2 \int_{\mathbb{T}^2} \bar{m}_1 g \, dx - \frac{t}{2\sqrt{Q-1}} \int_{\mathbb{T}^2} |\nabla^{\frac{1}{2}} \bar{m}_1|^2 \, dx. \end{aligned} \quad (1.109)$$

Proof. The lower bound for E in Lemma 1.5.1 implies a lower bound for the rescaled energy J

$$\begin{aligned} J(m) &= \frac{E(m(t, \ell \cdot, \ell \cdot)) - \ell^2 t}{\ell t \sqrt{Q-1}} \stackrel{(1.88)}{\geq} (1 - Ct^2) \int_{(0,1) \times \mathbb{T}^2} \left(\frac{1}{\ell \sqrt{Q-1}} |\nabla' m|^2 \right. \\ &\quad \left. + \frac{\ell}{t^2 \sqrt{Q-1}} |\partial_1 m|^2 + \ell \sqrt{Q-1} (m_2^2 + m_3^2) \right) dx \\ &\quad - \frac{2\ell}{\sqrt{Q-1}} \int_{\mathbb{T}^2} \bar{m}_1(x') h_{\text{ext}}(\ell x') \, dx - \frac{t}{2\sqrt{Q-1}} \int_{\mathbb{T}^2} |\nabla^{\frac{1}{2}} \bar{m}_1|^2 \, dx. \end{aligned} \quad (1.110)$$

We insert

$$\frac{\ell}{\sqrt{Q-1}} h_{\text{ext}}(\ell x') = g(x') \quad \text{and} \quad \varepsilon = \frac{1}{\ell \sqrt{Q-1}}$$

into (1.110) to obtain

$$\begin{aligned} J(m) &\stackrel{(1.88)}{\geq} (1 - Ct^2) \int_{(0,1) \times \mathbb{T}^2} \left(\varepsilon |\nabla' m|^2 + \frac{|\partial_1 m|^2}{\varepsilon t^2 (Q-1)} + \frac{1}{\varepsilon} (m_2^2 + m_3^2) \right) dx \\ &\quad - 2 \int_{\mathbb{T}^2} g \bar{m}_1 \, dx - \frac{t}{2\sqrt{Q-1}} \int_{\mathbb{T}^2} |\nabla^{\frac{1}{2}} \bar{m}_1|^2 \, dx. \end{aligned} \quad (1.111)$$

In view of (1.107) we may assume that

$$(1 - Ct^2) \left(\frac{1}{\varepsilon t^2 (Q-1)} - \varepsilon \right) \geq \frac{1}{2\varepsilon t^2 (Q-1)}. \quad (1.112)$$

Hence, applying Lemma 1.3.1 to the last term in (1.111) and inserting (1.112) we arrive at (1.108). The proof for the upper bound (1.109) is simpler and analogous to the arguments that led to (1.111). \square

1.6.1 Proof of Theorem 1.2.1

It is possible to invoke the lower bound for F on slices $\{x_1 = \text{const}\}$ to obtain the lower bound for the full (rescaled) energy J . However, we will not pursue this option. Instead, we apply the $H^{\frac{1}{2}}$ -bound of Lemma 1.3.1 directly and extend the arguments of the previous section. The reason is related to the fact that $C^\infty(\mathbb{T}^2 \times (0, 1); \mathbb{S}^2)$ is not dense in $H^1(\mathbb{T}^2 \times (0, 1); \mathbb{S}^2)$, which can be seen by considering $f(x) = \frac{x}{|x|}$ (see [8, 7, 41]). Hence, evaluating Sobolev functions on slices $\{x_1 = \text{const}\}$ and confirming that the constraint $|m| = 1$ still holds requires to use the precise representative of a Sobolev function and gets rather technical.

Proof of the lower bound and compactness in Theorem 1.2.1. Our starting point is the lower bound (1.108). It turns out to be more convenient to use the parameter $\varepsilon = \frac{1}{\ell\sqrt{Q-1}}$ instead of ℓ . We first note that for $\varepsilon < 1$ the last term in (1.108) may be estimated with the aid of (A.8) and (A.7) by

$$\begin{aligned} & \log \left(c_* \max \left\{ 1, \min \left\{ \frac{1}{\varepsilon \int_{\mathbb{T}^2} |\nabla m_1| \, dx}, \frac{1}{\varepsilon} \right\} \right\} \right) \int_{\mathbb{T}^2} |\nabla \overline{m}_1| \, dx \\ & \leq \log(c_*/\varepsilon) \int_{(0,1) \times \mathbb{T}^2} \left(\varepsilon |\nabla m|^2 + \frac{1}{\varepsilon} (m_2^2 + m_3^2) \right) \, dx. \end{aligned} \quad (1.113)$$

For Q and $(\ell_k, t_k, h_{\text{ext},k})$ satisfying (1.14), we abbreviate

$$\varepsilon_k := \frac{1}{\ell_k \sqrt{Q-1}} \rightarrow 0 \quad \text{and} \quad g_k := \frac{\ell_k}{\sqrt{Q-1}} h_{\text{ext},k}(\ell_k \cdot) \rightarrow g, \quad (1.114)$$

and note that

$$t_k^2 + \frac{t_k}{\sqrt{Q-1}} \xrightarrow{(1.14)} 0. \quad (1.115)$$

Inserting (1.14) and (1.113) – (1.115) into the lower bound (1.108), we deduce that for any $\gamma > 0$ and sufficiently large $k \geq k_0(\gamma)$, we have

$$\begin{aligned} J_k(m) & \geq \left(1 - \frac{\lambda}{\lambda_c} - \gamma \right) \int_{(0,1) \times \mathbb{T}^2} \left(\varepsilon_k |\nabla m|^2 + \frac{1}{\varepsilon_k} (m_2^2 + m_3^2) \right) \, dx \\ & \quad + \frac{1}{2\varepsilon_k t_k^2 (Q-1)} \int_{(0,1) \times \mathbb{T}^2} |\partial_1 m|^2 \, dx - 2 \int_{\mathbb{T}^2} \overline{m}_1 g_k \, dx. \end{aligned} \quad (1.116)$$

Note that (1.116) for $2\gamma \leq 1 - \frac{\lambda}{\lambda_c}$ and sufficiently large k implies

$$\int_{(0,1) \times \mathbb{T}^2} (m_2^2 + m_3^2) \, dx \lesssim \frac{\varepsilon_k}{(\lambda_c - \lambda)} (J_k(m) + 2\|g_k\|_{L^1}). \quad (1.117)$$

Using Poincaré's inequality and (1.116) for $\gamma < 1 - \frac{\lambda}{\lambda_c}$ again, we get

$$\begin{aligned} \int_{(0,1) \times \mathbb{T}^2} |m - \chi_{(0,1)} \bar{m}|^2 \, dx &\lesssim \int_{(0,1) \times \mathbb{T}^2} |\partial_1 m|^2 \, dx \\ &\stackrel{(1.116)}{\lesssim} \varepsilon_k t_k^2 (Q - 1) \left(\limsup_{k \rightarrow \infty} J_k(m) + 2\|g_k\|_{L^1} \right). \end{aligned} \quad (1.118)$$

Furthermore, applying (A.7) and (A.8) to (1.116) again implies the lower bound

$$J(m) \geq 2 \left(1 - \frac{\lambda}{\lambda_c} - \gamma \right) \int_{\mathbb{T}^2} |\nabla' \bar{m}_1| \, dx - 2 \int_{\mathbb{T}^2} \bar{m}_1 g_k \, dx. \quad (1.119)$$

In order to prove compactness, let $m^{(k)} \in H^1((0,1) \times \mathbb{T}^2; \mathbb{S}^2)$ with $\limsup_{k \rightarrow \infty} J(m_k) < \infty$. Since $\lambda < \lambda_c$ and $g_k \rightarrow g$ in $L^1(\mathbb{T}^2)$, inequality (1.119) implies a uniform bound on $\bar{m}_1^{(k)}$ in $BV(\mathbb{T}^2)$. A standard compactness argument implies that $\bar{m}_1^{(k)} \rightarrow \bar{m}_1$ in $L^1(\mathbb{T}^2)$ for a subsequence (not relabelled) and some $\bar{m}_1 \in BV(\mathbb{T}^2)$. We will now show that in fact $m^{(k)} \rightarrow \chi_{(0,1)} \bar{m}_1 e_1$ in $L^1((0,1) \times \mathbb{T}^2; \mathbb{R}^3)$. Indeed, the triangle inequality yields

$$\begin{aligned} \int_{(0,1) \times \mathbb{T}^2} |m^{(k)} - \chi_{(0,1)} \bar{m}_1 e_1| \, dx &\leq \int_{(0,1) \times \mathbb{T}^2} \left(|m_2^{(k)}|^2 + |m_3^{(k)}|^2 \right)^{\frac{1}{2}} \, dx \\ &+ \int_{(0,1) \times \mathbb{T}^2} |m_1^{(k)} - \chi_{(0,1)} \bar{m}_1^{(k)}| \, dx + \int_{\mathbb{T}^2} |\bar{m}_1^{(k)} - \bar{m}_1| \, dx, \end{aligned} \quad (1.120)$$

and we already know that the last term on the right hand side of (1.120) vanishes. Furthermore, the first term vanishes due to (1.117) and the second one due to (1.118) and (1.14). This completes the proof of the compactness statement.

The liminf inequality is easily obtained from the lower bound (1.119). Indeed, let $m^{(k)} \in H^1((0,1) \times \mathbb{T}^2; \mathbb{S}^2)$ with $m^{(k)} \rightarrow m$ in $L^1((0,1) \times \mathbb{T}^2)$. By Jensen's inequality, we also have $\bar{m}^{(k)} \rightarrow \bar{m}$ in $L^1(\mathbb{T}^2)$. By lower semicontinuity of the BV seminorm and since γ was arbitrary, we obtain from (1.119) in the limit

$$\liminf_{k \rightarrow \infty} J_k(m^{(k)}) \geq \left(1 - \frac{\lambda}{\lambda_c} \right) \int_{\mathbb{T}^2} |\nabla' \bar{m}_1| \, dx - 2 \int_{\mathbb{T}^2} \bar{m}_1 g \, dx.$$

□

It remains to prove the upper bound for the Γ -convergence. As it turns out, we may use the recovery sequence for the reduced energy F also for the full energy E (up to thickening).

Construction of the recovery sequence in Theorem 1.2.1. Let $\lambda \leq \lambda_c$ and $\bar{m} \in BV(\mathbb{T}^2; \{\pm e_1\})$. Furthermore, let $\bar{m}_\varepsilon \in H^1(\mathbb{T}^2; \mathbb{S}^2)$ denote the recovery sequence for $F_{\varepsilon, \lambda}$ from Lemma 1.4.3. With the notation (1.114) we set

$$m^{(k)}(x_1, x') := \chi_{(0,1)}(x_1) \bar{m}_{\varepsilon_k}(x') \quad \text{for } (x_1, x') \in (0, 1) \times \mathbb{T}^2$$

and claim that

$$\limsup_{k \rightarrow \infty} J_k(m^{(k)}) \leq J_*(\bar{m}).$$

Inserting the abbreviation $\lambda_k := \frac{t_k |\log(\varepsilon_k)|}{4\sqrt{Q-1}}$ into the upper bound (1.109), we obtain

$$\begin{aligned} J_k(m^{(k)}) &\leq (1 + Ct_k^2) \int_{\mathbb{T}^2} \left(\varepsilon_k |\nabla \bar{m}_{\varepsilon_k}|^2 + \frac{1}{\varepsilon_k} (\bar{m}_{\varepsilon_k,2}^2 + \bar{m}_{\varepsilon_k,3}^2) \right) dx \\ &\quad - \frac{2\lambda_k}{|\log \varepsilon_k|} \int_{\mathbb{T}^2} |\nabla^{\frac{1}{2}} \bar{m}_{\varepsilon_k,1}|^2 - 2 \int_{\mathbb{T}^2} g_k \bar{m}_{\varepsilon_k,1} dx \\ &= 2F_{\varepsilon_k, \lambda_k}[\bar{m}_{\varepsilon_k}] - 2 \int_{\mathbb{T}^2} g_k \bar{m}_{\varepsilon_k,1} dx \\ &\quad + Ct_k^2 \int_{\mathbb{T}^2} \left(\varepsilon_k |\nabla \bar{m}_{\varepsilon_k}|^2 + \frac{1}{\varepsilon_k} (\bar{m}_{\varepsilon_k,2}^2 + \bar{m}_{\varepsilon_k,3}^2) \right) dx. \end{aligned} \tag{1.121}$$

We have shown in Lemma 5.3 that

$$\int_{\mathbb{T}^2} \left(\varepsilon_k |\nabla \bar{m}_{\varepsilon_k}|^2 + \frac{1}{\varepsilon_k} (\bar{m}_{\varepsilon_k,2}^2 + \bar{m}_{\varepsilon_k,3}^2) \right) dx \rightarrow 2 \int_{\mathbb{T}^2} |\nabla \bar{m}_1| dx < \infty.$$

Since (1.14) implies $t_k \rightarrow 0$, $\lambda_k \rightarrow \lambda < \lambda_c$ and $g_k \rightarrow g$ in $L^1(\mathbb{T}^2)$, the claim follows upon applying Lemma 1.4.3 to (1.121)

$$\limsup_{k \rightarrow \infty} J_k[m^{(k)}] \leq 2F_{*, \lambda}(\bar{m}) - 2 \int_{\mathbb{T}^2} g \bar{m}_1 dx.$$

□

1.6.2 Proof of Theorem 1.2.2

Theorem 1.2.2. We begin with the proof of the lower bound for which we use (1.108) with $g = 0$. For sufficiently small δ , the regime (1.16) implies

$$Ct^2 + \frac{t}{\sqrt{Q-1}} \stackrel{(1.16)}{\lesssim} C\delta^2 + \delta \lesssim \delta. \quad (1.122)$$

Analogous to the argument that lead from (1.75) to (1.77), but now with (1.122) instead of (1.70), we reduce (1.108) to the case

$$\begin{aligned} J(m) \geq & \left(1 - Ct^2 - \frac{t}{\sqrt{Q-1}}\right) \int_{(0,1) \times \mathbb{T}^2} \left(\varepsilon |\nabla m|^2 + \frac{1}{\varepsilon}(m_2^2 + m_3^2)\right) dx \\ & + \frac{1}{2\varepsilon t^2(Q-1)} |\partial_1 m|^2 dx - \frac{t \log \left(c_* \frac{1}{\varepsilon \int_{\mathbb{T}^2} |\nabla \bar{m}_1| dx}\right)}{\pi \sqrt{Q-1}} \int_{\mathbb{T}^2} |\nabla \bar{m}_1| dx. \end{aligned} \quad (1.123)$$

Abbreviating the energetic cost for m to deviate from the optimal Bloch wall profile by

$$D_\varepsilon(m) := \int_{(0,1) \times \mathbb{T}^2} \left(\varepsilon |\nabla m|^2 + \frac{1}{\varepsilon}(1 - m_1^2)\right) dx - 2 \int_{\mathbb{T}^2} |\nabla \bar{m}_1| dx, \quad (1.124)$$

and inserting $\mu := \varepsilon e^{2\pi t^{-1}\sqrt{Q-1}} \int_{\mathbb{T}^2} |\nabla \bar{m}_1| dx$ and $c_{**} := c_* e^{2\pi(1+Ct\sqrt{Q-1})} \stackrel{(1.16)}{\sim} 1$ into the lower bound (1.123) we get

$$\begin{aligned} J(m) \geq & \left(1 - Ct^2 - \frac{t}{\sqrt{Q-1}}\right) D_\varepsilon(m) + \frac{1}{2\varepsilon t^2(Q-1)} \int_{(0,1) \times \mathbb{T}^2} |\partial_1 m|^2 dx \\ & - \frac{\log(c_{**}/\mu)}{\pi} \mu t \ell e^{-2\pi t^{-1}\sqrt{Q-1}}. \end{aligned} \quad (1.125)$$

Minimizing in $\mu > 0$ then yields the lower bound

$$J(m) \gtrsim -c_{**} t \ell e^{-2\pi t^{-1}\sqrt{Q-1}} \gtrsim -t \ell e^{-2\pi t^{-1}\sqrt{Q-1}}.$$

It remains to construct a sequence that achieves the optimal scaling. Let $m_{\varepsilon,N}$ denote the function constructed in Lemma 1.4.5 and define $m_{\varepsilon,N} := \chi_{(0,1)} \bar{m}_{\varepsilon,N}$. We insert (1.81) and (1.82) into (1.109) and use that (1.16) implies $t^2 \lesssim \frac{t}{\sqrt{Q-1}}$ to deduce

$$J(m_{\varepsilon,N}) \leq 4N \left(1 + Ct^2 - \frac{t \log(\frac{c}{2\varepsilon N})}{2\pi \sqrt{Q-1}}\right) \stackrel{(1.16)}{\leq} 4N \left(1 - \frac{t \log(\frac{\tilde{c}}{2\varepsilon N})}{2\pi \sqrt{Q-1}}\right) \quad (1.126)$$

for some universal $\tilde{c} > 0$. Optimizing in N leads to

$$N := 2 \left\lfloor \ell \sqrt{Q-1} \frac{e^{-2\pi t^{-1}\sqrt{Q-1}}}{K} \right\rfloor, \quad (1.127)$$

which satisfies $N \geq 2$ due to (1.16) and is hence admissible. Inserting (1.127) into (1.126), and taking $K \geq \frac{8}{\tilde{c}}$, we conclude that the function $m_{\varepsilon, N}$ indeed achieves the optimal scaling

$$J(m_{\varepsilon, N}) \lesssim -t\ell e^{-2\pi t^{-1}\sqrt{Q-1}}.$$

□

1.6.3 Proof of Proposition 1.2.3

Proposition 1.2.3. Let m satisfy (1.17). Then (1.122) and (1.125) imply $\mu \sim 1$ and hence (1.20)

$$\int_{\mathbb{T}^2} |\nabla \overline{m}_1| \, dx \sim \ell \sqrt{Q-1} e^{-2\pi t^{-1}\sqrt{Q-1}},$$

where here and throughout the rest of this proof, the constants associated with \lesssim, \gtrsim and \sim may depend on γ . In turn, inserting (1.17), (1.20) and (1.122) into (1.125) implies (1.21)

$$D_\varepsilon(m) \stackrel{(1.125)}{\lesssim} \frac{t}{\sqrt{Q-1}} \int_{\mathbb{T}^2} |\nabla \overline{m}_1| \, dx.$$

Furthermore, Poincaré's inequality, (1.125), (1.17) and $\mu \sim 1$ yield (1.18)

$$\begin{aligned} \int_{(0,1) \times \mathbb{T}^2} |m - \chi_{(0,1)} \overline{m}|^2 \, dx &\lesssim \int_{(0,1) \times \mathbb{T}^2} |\partial_1 m|^2 \, dx \\ &\stackrel{(1.125)}{\lesssim} t^3 \sqrt{Q-1} e^{-2\pi t^{-1}\sqrt{Q-1}}. \end{aligned}$$

Finally, we deduce (1.19) from (1.124), (1.20) and (1.21)

$$\begin{aligned} \int_{(0,1) \times \mathbb{T}^2} (m_2^2 + m_3^2) \, dx &\stackrel{(1.124)}{\lesssim} \varepsilon \left(\int_{\mathbb{T}^2} |\nabla \overline{m}_1| \, dx + D_\varepsilon(m) \right) \\ &\lesssim e^{-2\pi t^{-1}\sqrt{Q-1}}, \end{aligned}$$

which completes the proof. □

1.6.4 Proof of Theorem 1.2.4

Theorem 1.2.4. The proof is analogous to the proof of Theorem 1.2.7. □

Chapter 2

The critical scaling

In the previous chapter, we have identified the scaling of the minimal micromagnetic energy and the typical length scale of patterns that achieve it. In this section, we make a first step towards a finer analysis, which corresponds to the next order in the Γ -development. Our focus is on the case where the length of the unit-cell is much larger than, but still comparable to, the expected pattern size. This is a special case of the “critical scaling” in the previous chapter.

Due to difficulties related in part to the diffuse interfaces, we are unable to carry out such a program for the full micromagnetic energy. Instead, our analysis in this chapter proceeds in two steps.

As a first step, presented in section 2.1, we use heuristic arguments to reduce the full micromagnetic energy to the reduced energy $H_{\eta,\gamma}$, defined below. Roughly speaking, $H_{\eta,\gamma}$ is a (suitably rescaled) sharp interface version of $F_{\varepsilon,\lambda}$ (see (0.4)). In contrast to chapter 1, it is now crucial to exploit the natural regularization in the stray field energy (cf. Remark 1.1.1).

For the second step, we return to mathematical rigor. In section 2.2, we state the main results of this chapter including the Γ -convergence of $H_{\eta,\gamma}$ towards a nonlocal limit energy. All proofs are given in section 2.3.

We introduce the reduced energy $H_{\eta,\gamma} : L^1(\mathbb{T}^n) \rightarrow \mathbb{R} \cup \{+\infty\}$, given by

$$\begin{aligned} H_{\eta,\gamma}(u) = & \log\left(\frac{1}{\gamma\eta}\right) \int_{\mathbb{T}^n} |\nabla u| \, dx \\ & - \frac{1}{c_n} \int_{\mathbb{T}^n} \int_{\mathbb{R}^n \setminus B_\eta} \frac{|u(x+z) - u(x)|^2}{|z|^{n+1}} \, dz \, dx \end{aligned} \tag{2.1}$$

for $u \in BV(\mathbb{T}^n; \{-1, 1\})$ and $H_{\eta, \gamma}(u) = +\infty$ otherwise. Here, $c_n := \frac{2(n+1)\omega_{n+1}}{\pi}$ is a dimensional constant (in particular, $c_1 = 4$ and $c_2 = 8$). We are interested in the dependence of $H_{\eta, \gamma}$ on the positive parameters η and γ . Whereas η may be interpreted as the aspect ratio of the ferromagnetic film (more precisely of its unit-cell), the interpretation of γ is less obvious (see (2.6) below). It will turn out – as a consequence of our analysis – that γ is proportional to the total interfacial length of minimizers of $H_{\eta, \gamma}$. Thus $1/\gamma$ may be interpreted as the rescaled typical distance of neighboring domain walls.

2.1 Heuristic derivation of the reduced energy

In this section, we provide a heuristic derivation of the reduced energy $H_{\eta, \gamma}$ from the full micromagnetic energy (0.1). As in chapter 1, we assume periodicity in the film plane and consider the energy in a unit-cell $(0, t) \times \mathbb{T}_\ell^2$ with non-dimensionalized thickness t and length ℓ . We also keep the assumption that the parameter Q is larger than 1.

Moreover, we also assume that the magnetization is two-dimensional, i.e. constant in the direction normal to the film. Recall that in chapter 1, we have considered ultrathin films and *proved* in particular that the magnetization is (asymptotically) two-dimensional. While our proof does not extend to thicker films, experimental observations (see, e.g., [44, Chapter 5.6]) indicate that this property continues to hold also for films of “intermediate” thicknesses. In the following, we assume that t is sufficiently small such that the magnetization is approximately two-dimensional.

We begin to renormalize the energy as in section 1.1.1. However, this time we keep the full stray field energy which yields

$$\begin{aligned} \frac{E(m(\ell \cdot)) - \ell^2 t}{\ell t \sqrt{Q-1}} &= \int_{\mathbb{T}^2} \left(\frac{1}{\ell \sqrt{Q-1}} |\nabla \bar{m}|^2 + \ell \sqrt{Q-1} (\bar{m}_2^2 + \bar{m}_3^2) \right) dx \\ &\quad + \frac{\ell}{\sqrt{Q-1}} \sum_{k \in 2\pi\mathbb{Z}^2} (\sigma(t/\ell|k|) - 1) |\widehat{\bar{m}}_{1,k}|^2 \\ &\quad + \frac{\ell}{\sqrt{Q-1}} \sum_{k \in 2\pi\mathbb{Z}^2} (1 - \sigma(t/\ell|k|)) \left| \frac{k}{|k|} \cdot \widehat{\bar{m}}'_k \right|^2. \end{aligned} \quad (2.2)$$

We neglect the last term, which is reasonable because it is non-negative and vanishes for the Bloch wall constructions that we used to obtain the upper

bound in chapter 1. The term in the second line will be approximated by a regularized $H^{\frac{1}{2}}$ -norm. This is the content of the following Lemma.

Lemma 2.1.1 (Multiplier estimate). *There are universal constants $c_1, c_2 > 0$ such that for all $s > 0$ and any $f \in L^2(\mathbb{T}^2)$ we have*

$$\sum_{k \in 2\pi\mathbb{Z}^2} (1 - \sigma(s|k|)) |\widehat{f}_k|^2 \geq \frac{s}{8\pi} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2 \setminus B_{c_1 s}} \frac{|f(x+z) - f(x)|^2}{|z|^3} dz dx, \quad (2.3)$$

$$\sum_{k \in 2\pi\mathbb{Z}^2} (1 - \sigma(s|k|)) |\widehat{f}_k|^2 \leq \frac{s}{8\pi} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2 \setminus B_{c_2 s}} \frac{|f(x+z) - f(x)|^2}{|z|^3} dz dx. \quad (2.4)$$

A proof of Lemma 2.1.1 is provided in section 2.3.3. Although the values of c_1 and c_2 are not equal, this difference is expected to affect only the implicit constants in the scaling laws. We thus approximate the energy by

$$\begin{aligned} \frac{E(m(\ell \cdot)) - \ell^2 t}{\ell t \sqrt{Q-1}} &\approx \int_{\mathbb{T}^2} \left(\frac{1}{\ell \sqrt{Q-1}} |\nabla \overline{m}|^2 + \ell \sqrt{Q-1} (\overline{m}_2^2 + \overline{m}_3^2) \right) dx \\ &\quad - \frac{t}{8\pi \sqrt{Q-1}} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2 \setminus B_{t/\ell}} \frac{|\overline{m}_1(x+z) - \overline{m}_1(x)|^2}{|z|^3} dz dx. \end{aligned} \quad (2.5)$$

Furthermore, we introduce

$$\gamma := \min \left\{ \sqrt{Q-1}, \frac{1}{t} \right\} \ell e^{-2\pi \sqrt{Q-1} t^{-1}} \quad (2.6)$$

which we expect to correspond to the number of domains in a unit cell. Note that, using the rescaled Bloch wall thickness $\varepsilon = \frac{1}{\ell \sqrt{Q-1}}$ and the aspect ratio $\eta = \frac{t}{\ell}$, (2.6) may be expressed as

$$\frac{2\pi \sqrt{Q-1}}{t} = \log \left(\frac{1}{\gamma \max\{\eta, \varepsilon\}} \right). \quad (2.7)$$

Expressed in terms of ε , η and γ , the renormalized micromagnetic energy (2.5) reads

$$\begin{aligned} \frac{E(m(\ell \cdot)) - \ell^2 t}{\ell t \sqrt{Q-1}} &\approx \int_{\mathbb{T}^2} \left(\varepsilon |\nabla \overline{m}|^2 + \frac{1}{\varepsilon} (\overline{m}_2^2 + \overline{m}_3^2) \right) dx \\ &\quad - \frac{1}{4 \log \left(\frac{1}{\gamma \max\{\eta, \varepsilon\}} \right)} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2 \setminus B_\eta} \frac{|\overline{m}_1(x+z) - \overline{m}_1(x)|^2}{|z|^3} dz dx. \end{aligned} \quad (2.8)$$

Let us first discuss the case $\varepsilon \ll \eta$, where the $H^{\frac{1}{2}}$ -term is almost oblivious to the width of the diffuse interfaces on scale ε due to its regularization on scale

η . Thus, it seems appropriate to pass to a sharp interface description. For $\varepsilon \gtrsim \eta$, we have already explained in Remark 1.1.1 that the regularization on scale η in the $H^{\frac{1}{2}}$ -term is negligible compared to the regularization due to the exchange energy on the larger scale ε (by enforcing a domain wall thickness of ε which is the smallest scale of m in the two-dimensional system). A sharp interface description should therefore contain a regularization of the $H^{\frac{1}{2}}$ -term on scale $\max\{\eta, \varepsilon\}$ which leads to

$$\begin{aligned} \frac{E(m(\ell \cdot)) - \ell^2 t}{\ell t \sqrt{Q-1}} &\approx 2 \int_{\mathbb{T}^2} |\nabla \overline{m}_1| \, dx \\ &- \frac{1}{4 \log \left(\frac{1}{\gamma \max\{\eta, \varepsilon\}} \right)} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2 \setminus B_{\max\{\eta, \varepsilon\}}} \frac{|\overline{m}_1(x+z) - \overline{m}_1(x)|^2}{|z|^3} \, dz \, dx. \end{aligned} \quad (2.9)$$

The connection between $H_{\eta, \gamma}$ and the micromagnetic energy is then given by

$$\frac{E(m(\ell \cdot)) - \ell^2 t}{\pi \ell t^2} \approx H_{\max\{\eta, \varepsilon\}, \gamma}(\overline{m}_1), \quad (2.10)$$

where $\eta = \frac{t}{\ell}$, $\varepsilon = \frac{1}{\ell \sqrt{Q-1}}$ and γ is as in (2.6). This heuristic indicates that with additional work, the scaling law for the typical domain size (1.23) might be generalized to (in physical variables)

$$S \sim \max \left\{ \frac{1}{\sqrt{Q-1}}, \frac{T}{l_{\text{ex}}} \right\} l_{\text{ex}} e^{\frac{2\pi l_{\text{ex}} \sqrt{Q-1}}{T}} \quad (2.11)$$

for sufficiently thin film thicknesses T (such that the magnetization turns out to be approximately two-dimensional).

2.2 Main results

In this section, we return to mathematical rigor and study the asymptotic behavior of $H_{\eta, \gamma}$ for $\eta \rightarrow 0$ with $\gamma > 0$ fixed. Our main result is the derivation of a nonlocal Γ -limit summarized in the next theorem.

Theorem 2.2.1 (Γ -convergence and compactness). *Fix a sequence $(\eta_k, \gamma_k)_{k \in \mathbb{N}}$ with $\eta_k \rightarrow 0^+$ and $\gamma_k \rightarrow \gamma > 0$ and set $H_k := H_{\eta_k, \gamma_k}$ (see (2.1)). Then the following holds*

(i) **Compactness** *For every sequence $(u_k)_{k \in \mathbb{N}}$ in $L^1(\mathbb{T}^n)$ with*

$$\limsup_{k \rightarrow \infty} H_k(u_k) < \infty \quad (2.12)$$

there is $u \in BV(\mathbb{T}^n; \{-1, 1\})$ such that (upon passing to a subsequence) $u_k \rightarrow u$ in $L^1(\mathbb{T}^n)$.

(ii) **Γ -convergence** The sequence $(H_k)_{k \in \mathbb{N}}$ Γ -converges with respect to the $L^1(\mathbb{T}^n)$ topology towards

$$H_{*,\gamma}(u) := \sup \{H_{\eta,\gamma}(u) : \eta > 0\}. \quad (2.13)$$

Since $H_{*,\gamma}$ is not given explicitly, we analyze it in more detail. First, we study its dependence on the parameter γ .

Theorem 2.2.2. (Dependence on γ) Let $H_{*,\gamma}$ be given by (2.13). There are constants $c_1, C_1 > 0$ (which only depend on the dimension n) such that the minimal energy $\min H_{*,\gamma}$ satisfies

$$-C_1\gamma \leq \min_{u \in L^1} H_{*,\gamma}(u) \leq -c_1\gamma \quad \text{for all } \gamma \geq \frac{\pi}{2}. \quad (2.14)$$

Moreover, if $H_{*,\gamma}(u) \leq c \min H_{*,\gamma}$ for some $c > 0$, there are constants $c_2, C_2 > 0$ such that

$$c_2\gamma \leq \int_{\mathbb{T}^n} |\nabla u| \, dx \leq C_2\gamma. \quad (2.15)$$

The constants c_2, C_2 depend on c , but are independent of γ and u .

Theorem 2.2.2 confirms our heuristic interpretation of γ as the "the number of domains in a unit-cell".

In the following, we give an alternative expression for $H_{*,\gamma}$ for $n = 1$. It uses the fact that the sequence $\{H_{\eta,\gamma}(u)\}_{\eta>0}$ becomes constant and hence independent of η for sufficiently small η . We begin to introduce some notation: Let $x_0, x_1, \dots, x_{2N-1}$ be points in $[0, 1]$ with $0 = x_0 < x_1 < \dots < x_{2N-1}$. We set $d_k := x_k - x_{k-1}$ and $x_{k+2N\ell} = x_k$ for $\ell \in \mathbb{Z}$ and $0 \leq k < 2N$. We may now define the 1-periodic function $u : \mathbb{R} \rightarrow \{-1, 1\}$ with jumps at x_k by (see Figure 2.1)

$$u = -1 + 2 \left(\sum_{k \in \mathbb{Z}} \chi_{(x_{2k}, x_{2k+1})} \right). \quad (2.16)$$

Note that $H_{\eta,\gamma}(u) < \infty$ implies $u \in BV(\mathbb{T}^n; \{\pm 1\})$ and hence, for $n = 1$, every u with $H_{*,\gamma}(u) < \infty$ has the form (2.16).

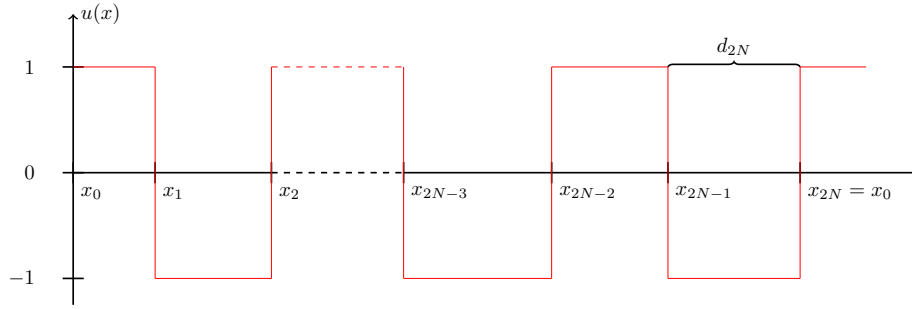


Figure 2.1: The function u as in (2.16).

Theorem 2.2.3 (Alternative representation of the energy). *For $n = 1$, u as in (2.16) and $\eta \leq \min d_k$ the energy $H_{\eta,\gamma}(u)$ may be written as*

$$H_{\eta,\gamma}(u) = \log \left(\frac{1}{e\gamma} \right) \int_{\mathbb{T}} |\nabla u| dx - 2 \sum_{j=0}^{\infty} \log \left(\prod_{k=1}^{2N} \frac{\left(\sum_{n=0}^{2j} d_{k+n} \right)^2}{\left(\sum_{n=0}^{2j-1} d_{k+n} \right) \left(\sum_{n=0}^{2j+1} d_{k+n} \right)} \right) \quad (2.17)$$

with the convention that for $j = 0$ the factors consisting of an empty sum are omitted. In the special case that $d_{2k-1} = \frac{1+s}{2N}$ and $d_{2k} = \frac{1-s}{2N}$ for some $s \in (-1, 1)$, we obtain

$$H_{\eta,\gamma}(u) = 4N \log \left(\frac{\pi N}{e\gamma \cos \left(\frac{\pi s}{2} \right)} \right). \quad (2.18)$$

For $n \geq 2$, let $\tilde{u} : \mathbb{T}^n \rightarrow \mathbb{R}$ with $\tilde{u}(x_1, x') = u(x_1)$ be the constant extension of u . Then identities (2.17) and (2.18) hold for \tilde{u} with “ \leq ” instead of “ $=$ ”.

The utility of the formula (2.17) is primarily due to the fact that its right hand side is independent of η .

2.3 Proofs

Our main tool is the estimate for the regularized homogeneous $H^{\frac{1}{2}}$ -norm (1.34), which we formulate here for arbitrary dimensions $n \in \mathbb{N}$.

Lemma 2.3.1. *For $n \in \mathbb{N}$, all $0 < r < R$ and all $u \in BV(\mathbb{T}^n; \mathbb{R})$, we have*

$$\int_{\mathbb{T}^n} \int_{B_R \setminus B_r} \frac{|u(x+z) - u(x)|^2}{|z|^{n+1}} dz dx \leq c_n \log(R/r) \|u\|_{L^\infty} \int_{\mathbb{T}^n} |\nabla u| dx, \quad (2.19)$$

where $c_n := \frac{2(n+1)\omega_{n+1}}{\pi}$ as in (2.1).

Lemma 2.3.1 may be proved literally as in the proof of Lemma 1.3.1, hence we do not repeat the proof here. A useful observation is that (2.19) is sharp for $n = 1$.

Lemma 2.3.2. *Let $n = 1$ and define $u \in BV(\mathbb{T}; \{-1, 1\})$ by $u = 1 - 2\chi_{(0, \frac{1}{2})}$ (extended periodically). Then u satisfies (2.19) with equality for all $0 < r < R < \frac{1}{2}$.*

Proof. Note that $c_1 = 4$. Hence, for all $0 < r < R < \frac{1}{2}$, we have

$$\begin{aligned} \int_{\mathbb{T}} \int_{B_R/B_r} \frac{|u(x+z) - u(x)|^2}{|z|^2} dz dx &= 4 \int_{\frac{1}{2}-R}^{\frac{1}{2}} \int_{\max\{\frac{1}{2}-x, r\}}^R \frac{4}{|z|^2} dz dx \\ &= 16 \int_{\frac{1}{2}-R}^{\frac{1}{2}} -\frac{1}{R} + \frac{1}{\max\{\frac{1}{2}-x, r\}} dx = 16 \left(-1 + \int_{\frac{1}{2}-R}^{\frac{1}{2}-r} \frac{1}{\frac{1}{2}-x} dx + 1 \right) \\ &= 16 \log(R/r) = 4 \log(R/r) \int_{\mathbb{T}^2} |\nabla u| dx. \end{aligned}$$

□

2.3.1 Proof of Theorem 2.2.1

The proof of Theorem 2.2.1 uses monotonicity and lower semi-continuity of $H_{\eta, \gamma}$, which we record in Lemma 2.3.3 below

Lemma 2.3.3. *For all $\gamma > 0$ and all $\eta \leq \frac{1}{\gamma}$, the functionals $H_{\eta, \gamma}$ and $H_{*, \gamma}$ are lower semi-continuous with respect to $L^1(\mathbb{T}^n)$ -convergence. Furthermore, $H_{\eta, \gamma}$ is monotone in $\eta > 0$, more precisely*

$$H_{\eta, \gamma}(u) \geq H_{\eta', \gamma}(u) \quad \text{for all } 0 < \eta \leq \eta' \text{ and all } u \in L^1(\mathbb{T}^n). \quad (2.20)$$

Proof. We first show that $H_{\eta, \gamma}$ is lower semi-continuous (l.s.c.). Since $\int_{\mathbb{T}^n} |\nabla u| dx$ is l.s.c. (see, e.g., [40, Theorem 1.9]), it remains to argue for the second term in $H_{\eta, \gamma}$. Let $u_k \rightarrow u$ in $L^1(\mathbb{T}^n)$. Without loss of generality, we may assume that $H_{\eta, \gamma}(u_\eta) < \infty$ and hence $u_\eta, u \in BV(\mathbb{T}^n; \{\pm 1\})$. Since $|u| \leq 1$, we get $u_k \rightarrow u$ in $L^2(\mathbb{T}^n)$. In particular

$$\int_{\mathbb{T}^n} |u_k(x+z) - u_k(x)|^2 dx \longrightarrow \int_{\mathbb{T}^n} |u(x+z) - u(x)|^2 dx \quad \text{for all } z \in \mathbb{R}^n.$$

By Fubini's theorem and dominated convergence, we conclude that

$$\int_{\mathbb{T}^n} \int_{\mathbb{R} \setminus B_\eta} \frac{|u_k(x+z) - u_k(x)|^2}{|z|^{n+1}} dz dx \rightarrow \int_{\mathbb{T}^n} \int_{\mathbb{R} \setminus B_\eta} \frac{|u(x+z) - u(x)|^2}{|z|^{n+1}} dz dx.$$

This shows that $H_{\eta,\gamma}$ is l.s.c for all $0 < \eta \leq 1/\gamma$. We conclude that the pointwise supremum of the l.s.c. functions $H_{*,\gamma} = \sup_{\eta \in (0,1/\gamma)} H_{\eta,\gamma}$ is also l.s.c.. Monotonicity of $H_{\eta,\gamma}$ is a straightforward consequence of (2.19). Indeed, for $\eta' \geq \eta > 0$, we have

$$\begin{aligned} H_{\eta,\gamma}(u) - H_{\eta',\gamma}(u) &= \log(\eta'/\eta) \int_{\mathbb{T}^n} |\nabla u| \, dx \\ &\quad - \frac{1}{c_n} \int_{\mathbb{T}^n} \int_{B_{\eta'} \setminus B_\eta} \frac{|u(x+z) - u(x)|^2}{|z|^{n+1}} \, dz \, dx \stackrel{(2.19)}{\geq} 0. \end{aligned}$$

□

Lemma 2.3.4. *Let $\eta > 0$. Then the energy $H_{\eta,\gamma}$ satisfies the lower bound*

$$H_{\eta,\gamma}(u) \geq \log\left(\frac{1}{R\gamma}\right) \int_{\mathbb{T}^n} |\nabla u| \, dx - \frac{4n\omega_n}{c_n R} \quad (2.21)$$

for all $u \in BV(\mathbb{T}^n; \{\pm 1\})$, all $\gamma > 0$ and all $R \geq \eta > 0$.

Proof. Since $R \geq \eta$, (2.20) and an elementary estimate imply

$$\begin{aligned} H_{\eta,\gamma}(u) &\geq \log\left(\frac{1}{R\gamma}\right) \int_{\mathbb{T}^n} |\nabla u| \, dx - \frac{1}{c_n} \int_{\mathbb{T}^n} \int_{\mathbb{R}^n \setminus B_R} \frac{|u(x+z) - u(x)|^2}{|z|^{n+1}} \, dz \, dx \\ &\geq \log\left(\frac{1}{R\gamma}\right) \int_{\mathbb{T}^n} |\nabla u| \, dx - \frac{4n\omega_n}{c_n R} \quad \text{for all } R \geq \eta > 0. \end{aligned}$$

□

We turn to the proof of Theorem 2.2.1.

Proof of Theorem 2.2.1. We begin with the proof of compactness. Using (2.21) with $R = \frac{1}{2\gamma_k}$ yields

$$\begin{aligned} \log(2) \int_{\mathbb{T}^n} |\nabla u_k| \, dx &\stackrel{(2.21)}{\leq} H_{\frac{1}{2\gamma_k}, \gamma_k}(u_k) + \frac{8n\omega_n}{c_n} \gamma_k \\ &\lesssim \limsup_{k \rightarrow \infty} H_k(u_k) + \gamma < \infty. \end{aligned} \quad (2.22)$$

Hence, precompactness of $(u_k)_{k \in \mathbb{N}}$ follows from compactness of the embedding $BV \hookrightarrow L^1$ (see, e.g., [40, Thm. 1.19]).

We turn to the proof of the liminf inequality. Let $u_k \rightarrow u$ in $L^1(\mathbb{T}^n)$. Without loss of generality, we may assume that

$$\liminf_{k \rightarrow \infty} H_k(u_k) = \lim_{k \rightarrow \infty} H_k(u_k) < \infty \quad (2.23)$$

and hence $(u_k)_{k \in \mathbb{N}}$ is bounded in $BV(\mathbb{T}^n; \{\pm 1\})$ (see (2.22)). By monotonicity of $H_{\eta, \gamma}$, we get

$$H_k(u_k) \geq H_{\eta', \gamma_k}(u_k) = H_{\eta', \gamma}(u_k) + \log \left(\frac{\gamma}{\gamma_k} \right) \int_{\mathbb{T}^n} |\nabla u_k| dx \quad \text{for } \eta_k \leq \eta'.$$

By lower semi-continuity of $H_{\eta', \gamma}$, and boundedness of $\|u_k\|_{BV}$ this implies

$$\liminf_{k \rightarrow \infty} H_k(u_k) \geq H_{\eta', \gamma}(u) \quad \text{for all } \eta' < \frac{1}{\gamma}. \quad (2.24)$$

Taking the supremum over $0 < \eta' < \frac{1}{\gamma}$ yields the liminf inequality. Due to the monotonicity of $H_{\eta, \gamma}$ the limsup inequality is easily obtained for the constant sequence $u_k = u$. \square

Proof of Theorem 2.2.2. The lower bound on the minimal energy follows by monotonicity of $H_{\eta, \gamma}$ in η and (2.21) with $R = \frac{1}{\gamma}$

$$H_{*, \gamma}(u) \geq H_{\frac{1}{\gamma}, \gamma}(u) \geq -\frac{4n\omega_n}{c_n} \gamma. \quad (2.25)$$

The upper bound is based on a one dimensional configuration u_{2N} with $2N$ equidistant walls. By Theorem 2.2.3, we obtain for all $n \geq 1$

$$H_{\eta, \gamma}(u_{2N}) \leq 4N \log \left(\frac{\pi N}{e\gamma} \right) \quad \text{for all } \eta \leq \frac{1}{2N}. \quad (2.26)$$

Taking N to be the smallest integer greater than $\frac{\gamma}{\pi}$ yields

$$\frac{1}{e} \leq \frac{\pi N}{e\gamma} \leq \frac{2}{e} \quad \text{for } \gamma \geq \frac{\pi}{2} \quad (2.27)$$

and hence

$$\min_{u \in L^1} H_{*, \gamma}(u) \leq H_{*, \gamma}(u_{2N}) \leq \frac{8\gamma}{\pi} \log \left(\frac{2}{e} \right) \quad \text{for all } \gamma \geq \frac{\pi}{2}. \quad (2.28)$$

Moreover, if $H_{*, \gamma}(u) \leq -c\gamma$ holds then (2.21) with $R = \frac{1}{\int_{\mathbb{T}^n} |\nabla u| dx}$ yields

$$-c\gamma \geq H_{*, \gamma}(u) \geq \left(\log \left(\frac{\int_{\mathbb{T}^n} |\nabla u| dx}{\gamma} \right) - \frac{4n\omega_n}{c_n} \right) \int_{\mathbb{T}^n} |\nabla u| dx \quad (2.29)$$

$$\geq \log \left(\frac{\int_{\mathbb{T}^n} |\nabla u| dx}{\gamma \kappa_n} \right) \int_{\mathbb{T}^n} |\nabla u| dx, \quad (2.30)$$

where we have abbreviated $\kappa_n = e^{\frac{4n\omega_n}{c_n}}$. Since $x \log(x) \leq -c$ and $x \in (0, \infty)$ implies $x \sim 1$, the claim follows. \square

2.3.2 Proof of Theorem 2.2.3

We turn to the proof of Theorem 2.2.3, which is mainly a direct computation of the integral. However, a subtle point is that the infinite sum which is obtained by evaluating the integral is only conditionally convergent if the terms involving a fixed pair (x_i, x_k) (see notation (2.16)) are computed individually and then summed over i and k . Instead, we will always consider certain 4-tuples of such points, because the associated sum turns out to be absolutely convergent.

Proof of Theorem 2.2.3. We rewrite the nonlocal part of the energy

$$\int_{\mathbb{T}} \int_{\mathbb{R} \setminus B_\eta} \frac{|u(x+z) - u(x)|^2}{z^2} dz dx = \sum_{k=1}^{2N} \int_{x_{k-1}}^{x_k} \int_{\mathbb{R} \setminus B_\eta} \frac{|u(x+z) - u(x)|^2}{z^2} dz dx.$$

Since u is constant on (x_{k-1}, x_k) , we get (inserting (2.16))

$$\begin{aligned} & \int_{x_{k-1}}^{x_k} \int_{\mathbb{R} \setminus B_\eta} \frac{|u(x+z) - u(x)|^2}{z^2} dz dx \\ &= 4 \sum_{j \in \mathbb{Z}} \int_{x_{k-1}}^{x_k} \int_{\mathbb{R} \setminus B_\eta} \frac{\chi_{(x_{k+2j}, x_{k+2j+1})}(x+z)}{z^2} dz dx \\ &= 4 \int_{x_{k-1}}^{x_k} \int_{\mathbb{R} \setminus B_\eta} \frac{\chi_{(x_k, x_{k+1})}(x+z)}{z^2} dz dx \\ & \quad + 4 \int_{x_{k-1}}^{x_k} \int_{\mathbb{R} \setminus B_\eta} \frac{\chi_{(x_{k-2}, x_{k-1})}(x+z)}{z^2} dz dx \\ & \quad + 4 \sum_{j=1}^{\infty} \int_{x_{k-1}}^{x_k} \int_{\mathbb{R} \setminus B_\eta} \frac{\chi_{(x_{k+2j}, x_{k+2j+1})}(x+z)}{z^2} dz dx \\ & \quad + 4 \sum_{j=1}^{\infty} \int_{x_{k-1}}^{x_k} \int_{\mathbb{R} \setminus B_\eta} \frac{\chi_{(x_{k-2(j+1)}, x_{k-2j-1})}(x+z)}{z^2} dz dx. \end{aligned} \tag{2.31}$$

Evaluating the integrals on the right hand side exploiting $\eta \leq \min d_k$, we

obtain

$$\begin{aligned}
& \int_{x_{k-1}}^{x_k} \int_{\mathbb{R} \setminus B_\eta} \frac{|u(x+z) - u(x)|^2}{z^2} dz dx \\
&= 4 + 4 \log \left(\frac{(x_k - x_{k-1})(x_{k+1} - x_k)}{\eta(x_{k+1} - x_{k-1})} \right) \\
&\quad + 4 + 4 \log \left(\frac{(x_k - x_{k-1})(x_k - x_{k-2})}{\eta(x_k - x_{k-2})} \right) \\
&\quad + 4 \sum_{j=1}^{\infty} \log \left(\frac{(x_{k+2j} - x_{k-1})(x_{k+2j+1} - x_k)}{(x_{k+2j+1} - x_{k-1})(x_{k+2j} - x_k)} \right) \\
&\quad + 4 \sum_{j=1}^{\infty} \log \left(\frac{(x_k - x_{k-2j-1})(x_{k-1} - x_{k-2j-2})}{(x_k - x_{k-2j-2})(x_{k-1} - x_{k-2j-1})} \right).
\end{aligned} \tag{2.32}$$

Note that the infinite sums converge absolutely because all terms have the same sign. Furthermore, the first two log-terms can be considered as the $j = 0$ terms from the third and forth line except for the vanishing factors $x_{k+2j} - x_k$ and $x_{k+2j-1} - x_{k-1}$. With the convention that vanishing factors are omitted, we get

$$\begin{aligned}
& \int_{x_{k-1}}^{x_k} \int_{\mathbb{R} \setminus B_\eta} \frac{|u(x+z) - u(x)|^2}{z^2} dz dx = 8(1 + \log(1/\eta)) \\
&\quad + 4 \sum_{j=0}^{\infty} \log \left(\frac{(x_{k+2j} - x_{k-1})(x_{k+2j+1} - x_k)}{(x_{k+2j+1} - x_{k-1})(x_{k+2j} - x_k)} \right) \\
&\quad + 4 \sum_{j=0}^{\infty} \log \left(\frac{(x_k - x_{k-2j-1})(x_{k-1} - x_{k-2j-2})}{(x_k - x_{k-2j-2})(x_{k-1} - x_{k-2j-1})} \right).
\end{aligned} \tag{2.33}$$

Inserting $d_k = x_k - x_{k-1}$ into (2.33) and summing over k , we get

$$\begin{aligned}
& \int_{\mathbb{T}} \int_{\mathbb{R} \setminus B_\eta} \frac{|u(x+z) - u(x)|^2}{z^2} dz dx = 16N(1 + \log(1/\eta)) \\
&\quad + 4 \sum_{j=0}^{\infty} \log \left(\prod_{k=1}^{2N} \frac{\left(\sum_{n=0}^{2j} d_{k+n} \right) \left(\sum_{n=0}^{2j} d_{k+1+n} \right)}{\left(\sum_{n=0}^{2j-1} d_{k+n} \right) \left(\sum_{n=0}^{2j+1} d_{k+n} \right)} \right) \\
&\quad + 4 \sum_{j=0}^{\infty} \log \left(\prod_{k=1}^{2N} \frac{\left(\sum_{n=0}^{2j} d_{k-n} \right) \left(\sum_{n=0}^{2j} d_{k-1-n} \right)}{\left(\sum_{n=0}^{2j+1} d_{k-n} \right) \left(\sum_{n=0}^{2j-1} d_{k-1-n} \right)} \right),
\end{aligned} \tag{2.34}$$

where for $j = 0$ the factors consisting of empty sums are omitted. By period-

icity of the d_k , this turns into

$$\begin{aligned} \int_{\mathbb{T}} \int_{\mathbb{R} \setminus B_\eta} \frac{|u(x+z) - u(x)|^2}{z^2} dz dx &= 16N(1 + \log(1/\eta)) \\ &+ 8 \sum_{j=0}^{\infty} \log \left(\prod_{k=1}^{2N} \frac{\left(\sum_{n=0}^{2j} d_{k+n} \right)^2}{\left(\sum_{n=0}^{2j-1} d_{k+n} \right) \left(\sum_{n=0}^{2j+1} d_{k+n} \right)} \right). \end{aligned} \quad (2.35)$$

Since $\int_{\mathbb{T}} |\nabla u| dx = 4N$, inserting (2.35) into the definition of $H_{\eta,\gamma}$ yields the claim

$$\begin{aligned} H_{\eta,\gamma}(u) &= \log \left(\frac{1}{\eta\gamma} \right) \int_{\mathbb{T}} |\nabla u| dx - \frac{1}{4} \int_{\mathbb{T}} \int_{\mathbb{R} \setminus B_\eta} \frac{|u(x+z) - u(x)|^2}{z^2} dz dx \\ &= \log \left(\frac{1}{e\gamma} \right) \int_{\mathbb{T}} |\nabla u| dx - 2 \sum_{j=0}^{\infty} \log \left(\prod_{k=1}^{2N} \frac{\left(\sum_{n=0}^{2j} d_{k+n} \right)^2}{\left(\sum_{n=0}^{2j-1} d_{k+n} \right) \left(\sum_{n=0}^{2j+1} d_{k+n} \right)} \right). \end{aligned}$$

For the special case $d_{2k-1} = \frac{1+s}{2N}$ and $d_{2k} = \frac{1-s}{2N}$ the second term simplifies to

$$\begin{aligned} &2 \sum_{j=0}^{\infty} \log \left(\prod_{k=1}^{2N} \frac{\left(\sum_{n=0}^{2j} d_{k+n} \right)^2}{\left(\sum_{n=0}^{2j-1} d_{k+n} \right) \left(\sum_{n=0}^{2j+1} d_{k+n} \right)} \right) \\ &= 4N \log \left(\frac{(1-s^2)}{4N} \prod_{j=1}^{\infty} \frac{(2j+1+s)(2j+1-s)}{(2j)(2j+2)} \right). \end{aligned} \quad (2.36)$$

With the aid of a Sine product formula (see, e.g., [1, p.75]), we obtain the identity

$$\begin{aligned} (1-s^2) \prod_{j=1}^{\infty} \frac{(2j+1+s)(2j+1-s)}{(2j)(2j+2)} &= 2(1-|s|) \prod_{j=1}^{\infty} \left(1 - \frac{(1-|s|)^2}{(2j)^2} \right) \\ &= \frac{4}{\pi} \sin \left(\frac{\pi}{2} (1-|s|) \right) = \frac{4}{\pi} \cos \left(\frac{\pi s}{2} \right). \end{aligned} \quad (2.37)$$

Inserting (2.36) and (2.37) into (2.17) yields the claim

$$H_{\eta,\gamma}(\tilde{u}_N) = -4N \left(1 + \log(\gamma) + \log \left(\frac{\cos \left(\frac{\pi s}{2} \right)}{\pi N} \right) \right) = 4N \log \left(\frac{\pi N}{e\gamma \cos \left(\frac{\pi s}{2} \right)} \right).$$

For $n \geq 2$ note that since \tilde{u} depends only on x_1 we have

$$\begin{aligned}
& \int_{\mathbb{T}^n} \int_{\mathbb{R}^n \setminus B_r} \frac{|\tilde{u}(x+z) - \tilde{u}(x)|^2}{|z|^2} dz dx \\
& \geq \int_{\mathbb{T}^n} \int_{\mathbb{R}^{n-1} \times (\mathbb{R} \setminus (-r, r))} \frac{|\tilde{u}(x+z) - \tilde{u}(x)|^2}{|z|^2} dz dx \\
& \geq \int_{\mathbb{T}} \int_{\mathbb{R}^{n-1} \times (\mathbb{R} \setminus (-r, r))} \frac{|u(x_1+z_1) - u(x_1)|^2}{(|z_1|^2 + |z'|^2)^{\frac{n+1}{2}}} dz_1 dz' dx_1 \\
& \stackrel{*}{\geq} \frac{c_n}{c_1} \int_{\mathbb{T}} \int_{\mathbb{R} \setminus (-r, r)} \frac{|u(x_1+z_1) - u(x_1)|^2}{|z_1|^2} dz_1 dx_1
\end{aligned} \tag{2.38}$$

where (at $*$) we have used the integral identity

$$\begin{aligned}
& \int_{\mathbb{R}^{n-1}} \frac{1}{(|z_1|^2 + |z'|^2)^{\frac{n+1}{2}}} dz' = \mathcal{H}^{n-2}(\mathbb{S}^{n-2}) \int_0^\infty \frac{1}{(|z_1|^2 + r^2)^{\frac{n+1}{2}}} r^{n-2} dr \\
& = \frac{\mathcal{H}^{n-2}(\mathbb{S}^{n-2})}{(n-1)|z_1|^2} = \frac{(n+1)\omega_{n+1}}{2\pi|z_1|^2} = \frac{c_n}{c_1|z_1|^2}.
\end{aligned}$$

□

2.3.3 Proof of Lemma 2.1.1

Proof of Lemma 2.1.1. We first show that there are universal constants $c_1, c_2 > 0$ such that

$$1 - \sigma(t|\xi|) \geq \frac{t}{8\pi} \int_{\mathbb{R}^2 \setminus B_{c_1 t}} \frac{|1 - e^{i\xi \cdot z}|^2}{|z|^3} dz, \tag{2.39}$$

$$1 - \sigma(t|\xi|) \leq \frac{t}{8\pi} \int_{\mathbb{R}^2 \setminus B_{c_2 t}} \frac{|1 - e^{i\xi \cdot z}|^2}{|z|^3} dz, \tag{2.40}$$

for all $t > 0$. To simplify the notation, we introduce $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, defined by

$$g(s) := \frac{1}{2\pi} \int_{\mathbb{R}^2 \setminus B_1} \frac{\sin^2(sz_1/2)}{|z|^3} dz. \tag{2.41}$$

By rotational symmetry and scaling we may write

$$\begin{aligned}
\frac{t}{8\pi} \int_{\mathbb{R}^2 \setminus B_{ct}} \frac{|1 - e^{i\xi \cdot z}|^2}{|z|^3} dz &= \frac{t}{8\pi} \int_{\mathbb{R}^2 \setminus B_{ct}} \frac{|1 - e^{i|\xi|z_1}|^2}{|z|^3} dz \\
\frac{1}{8\pi c} \int_{\mathbb{R}^2 \setminus B_1} \frac{|1 - e^{i|\xi|ctz_1}|^2}{|z|^3} dz &= \frac{1}{2\pi c} \int_{\mathbb{R}^2 \setminus B_1} \frac{\sin^2(|\xi|ctz_1/2)}{|z|^3} dz \\
&= \frac{1}{c} g(ct|\xi|).
\end{aligned} \tag{2.42}$$

We thus have to show that there are c_1, c_2 such that

$$\frac{1}{c_2}g(c_2s) \leq 1 - \sigma(s) \leq \frac{1}{c_1}g(c_1s) \quad \text{for all } s \in (0, \infty). \quad (2.43)$$

In order to prove (2.43), we investigate the asymptotic behavior of $1 - \sigma$ and g for $s \rightarrow 0$ and $s \rightarrow \infty$. Evaluation of the limit $s \rightarrow \infty$ and a Taylor expansion at 0 yield

$$\lim_{s \rightarrow \infty} 1 - \sigma(s) = 1 \quad \text{and} \quad 1 - \sigma(s) = s/2 - s^2/6 + s^3/24 + O(s^4). \quad (2.44)$$

Moreover, we conclude from $\sin^2(sz_1/2) \rightharpoonup^* \frac{1}{2}$ in $L^\infty(\mathbb{R}^2)$ as $s \rightarrow \infty$ that

$$\lim_{s \rightarrow \infty} g(s) = \frac{1}{4\pi} \int_{\mathbb{R}^2 \setminus B_1} \frac{1}{|z|^3} dz = \frac{1}{2}. \quad (2.45)$$

Furthermore, expressing the integral (2.41) in polar coordinates and substituting $\rho = \frac{sr|\cos \theta|}{2}$, we get

$$\begin{aligned} g(s) &= \frac{1}{2\pi} \int_1^\infty \int_0^{2\pi} \frac{\sin^2(sr|\cos \theta|/2)}{r^2} d\theta dr \\ &= \frac{s}{4\pi} \int_0^{2\pi} |\cos \theta| \int_{\frac{s|\cos \theta|}{2}}^\infty \frac{\sin^2(\rho)}{\rho^2} d\rho d\theta. \end{aligned} \quad (2.46)$$

Inserting the identity $\int_0^\infty \frac{\sin^2(\rho)}{\rho^2} d\rho = \frac{\pi}{2}$ into (2.46), we arrive at

$$\begin{aligned} g(s) &= \frac{s}{4\pi} \int_0^{2\pi} |\cos \theta| \left(\frac{\pi}{2} - \int_0^{\frac{s|\cos \theta|}{2}} \frac{\sin^2(\rho)}{\rho^2} d\rho \right) d\theta \\ &= \frac{s}{2} - \frac{s}{4\pi} \int_0^{2\pi} |\cos \theta| \int_0^{\frac{s|\cos \theta|}{2}} \frac{\sin^2(\rho)}{\rho^2} d\rho d\theta. \end{aligned} \quad (2.47)$$

Since $\frac{\sin^2(\rho)}{\rho^2} \sim 1$ for $\rho \leq \frac{3}{2}$, (2.47) implies

$$g(s) - \frac{s}{2} \sim -s^2 \quad \text{for } s \leq 3. \quad (2.48)$$

We begin to prove the first inequality in (2.43) by showing that $c(1 - \sigma(\frac{s}{c})) - g(s)$ is non-negative. Indeed, we find

$$\begin{aligned} c(1 - \sigma(\frac{s}{c})) - g(s) &\stackrel{(2.44), (2.48)}{\geq} \frac{s}{2} - \frac{s^2}{6c} - \frac{s}{2} + Cs^2 \\ &= (C - \frac{1}{6c})s^2 \geq 0 \text{ for } s \leq 3 \text{ and } c \text{ sufficiently small} \end{aligned} \quad (2.49)$$

where C denotes the universal constant implicitly contained in (2.48). For $s \geq 3$ we have due to the monotonicity of σ

$$c(1 - \sigma(\frac{s}{c})) = c(1 - \sigma(\frac{3}{c})) \stackrel{(2.44)}{\geq} \frac{3}{2} - O(\frac{1}{c}). \quad (2.50)$$

In view of (2.45) and the monotonicity of $1 - \sigma$, we may assume that there is $s^* > 3$ with $g(s^*) = \max_{s \geq 3} g(s) \geq \frac{3}{2}$ (if not, the proof of the first inequality in (2.43) is complete). On the compact interval $[3, s^*]$ the strict inequality $g(s) < \frac{s}{2}$ turns into

$$g(s) \leq \frac{s}{2} - \delta \quad \text{for all } s \in [3, s^*] \quad (2.51)$$

and for some sufficiently small $\delta > 0$. Hence, upon taking c sufficiently large, we have

$$c(1 - \sigma(\frac{s}{c})) \stackrel{(2.44)}{\geq} \frac{s}{2} - O(\frac{s^2}{c}) \geq \frac{s}{2} - \delta \stackrel{(2.50)}{\geq} g(s) \quad \text{for all } s \in [3, s^*]. \quad (2.52)$$

On the remaining interval $[s^*, \infty)$, the claim follows from the monotonicity of $1 - \sigma$. We turn to the proof of the second inequality in (2.43). For sufficiently small c , we find

$$\begin{aligned} \frac{1}{c}g(cs) - (1 - \sigma(s)) &\stackrel{(2.44), (2.48)}{\geq} s/2 - Ccs^2 - (s/2 - s^2/6 + s^3/24) \\ &= \left(\frac{1}{6} - Cc - \frac{s}{24}\right)s^2 \geq 0 \quad \text{for } s \leq 3. \end{aligned} \quad (2.53)$$

It remains to show the inequality for $s \geq 3$. Since g is strictly positive on $(0, \infty)$ and by (2.45) and (2.48) we have for sufficiently small $\tilde{c} > 0$ that

$$\inf_{s \geq s_0} g(s) = g(s_0) \quad \text{for all } 0 < s_0 \leq \tilde{c}. \quad (2.54)$$

We conclude that for $3c \leq \tilde{c}$, we have

$$\inf_{s \geq 3} \frac{1}{c}g(cs) \stackrel{(2.54)}{=} \frac{1}{c}g(3c) \stackrel{(2.48)}{\geq} \frac{3}{2}(1 - O(c)) \geq 1 = \sup_{s \in (0, \infty)} 1 - \sigma(s), \quad (2.55)$$

which completes the proof of (2.43).

We turn to the proof of (2.3), the proof of (2.4) is essentially the same. By a density argument, we may assume that $f \in C_c^\infty(\mathbb{T}^2)$. The multiplier estimate (2.39) yields

$$\sum_{k \in 2\pi\mathbb{Z}^2} (1 - \sigma(t|k|)) |\widehat{f}_k|^2 \geq \sum_{k \in 2\pi\mathbb{Z}^2} \left(\frac{t}{8\pi} \int_{\mathbb{R}^2 \setminus B_{c_1 t}} \frac{|1 - e^{i\xi \cdot z}|^2}{|z|^3} dz \right) |\widehat{f}_k|^2.$$

Changing the order of integration and summation on the right hand side by means of Fubini's Theorem and inserting the identity

$$\sum_{k \in 2\pi\mathbb{Z}^2} |1 - e^{i\xi \cdot z}|^2 |\widehat{f}_k|^2 = \int_{\mathbb{R}^2} |f(x+z) - f(x)|^2 d\xi \quad (2.56)$$

yields the claim

$$\sum_{k \in 2\pi\mathbb{Z}^2} (1 - \sigma(t|k|)) |\widehat{f}_k|^2 \geq \frac{t}{8\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2 \setminus B_{c_1 t}} \frac{|f(x+z) - f(x)|^2}{|z|^3} dz d\xi. \quad (2.57)$$

□

Chapter 3

Optimal shape of a single domain

In this chapter, we are interested in the shape and the regularity of magnetic domains in a ferromagnetic film exposed to an external magnetic field close to saturation. Based on the results in [56], we introduce a simple model for a single magnetic domain in such films: a subset of \mathbb{R}^3 that minimizes the sum of its surface area and stray field energy among competitors of the same volume. In the following, we first give a precise definition of our model and introduce the necessary notation. The analysis of this minimization problem is the main topic of this chapter. The relation between the full micromagnetic energy and our prototypical model is discussed in section 3.1.

For $n \geq 2$ we define the energy of a measurable set $\Omega \subset \mathbb{R}^n$ by

$$\mathcal{E}(\Omega) = \mathcal{P}(\Omega) + \int_{\mathbb{R}^n} |\nabla \Phi_\Omega|^2 dx, \quad (3.1)$$

where $\mathcal{P}(\Omega)$ denotes the perimeter of Ω . The potential $\Phi_\Omega : \mathbb{R}^n \rightarrow \mathbb{R}$ is defined as the unique distributional solution of

$$\Delta \Phi_\Omega = \partial_1 \chi_\Omega, \quad \lim_{|x| \rightarrow \infty} \Phi_\Omega(x) = 0, \quad (3.2)$$

where χ_Ω denotes the characteristic function of Ω . We study the problem of minimizing (3.1) subject to a volume constraint, i.e. we minimize the energy \mathcal{E} over the admissible class

$$\mathcal{C}_V := \{\Omega \subset \mathbb{R}^n : \Omega \text{ is of finite perimeter and } |\Omega| = V\}. \quad (3.3)$$

The second term in (3.1) can be understood as the dipolar self-energy of a uniform dipole density on $\Omega \subset \mathbb{R}^n$, proportional to e_1 . Our focus is on the competing effects of interfacial versus dipolar energy. We study a full-space problem, there are no geometric constraints which might alter the nature of the problem. We only prescribe the volume of the set Ω and thus the relative strength of the two energy contributions.

We abbreviate the nonlocal term in (3.1) by

$$\mathcal{N}(\Omega) := \int_{\mathbb{R}^n} |\nabla \Phi_\Omega|^2 dx$$

and remark that it amounts to the squared \dot{H}^{-1} -norm of $\partial_1 \chi_\Omega$. An important quantity for our analysis is the *interaction energy* of two disjoint sets $F, G \subset \mathbb{R}^n$, given by

$$I(F, G) := \mathcal{N}(F \cup G) - \mathcal{N}(F) - \mathcal{N}(G) = 2 \int_{\mathbb{R}^n} \nabla \Phi_F \cdot \nabla \Phi_G dx, \quad (3.4)$$

where Φ_F satisfies (3.2). It is instructive to express the interaction energy by means of the strongly singular Calderón–Zygmund kernel

$$\partial_1^2 \Gamma(z) = \frac{|z|^2 - nz_1^2}{n\omega_n |z|^{n+2}}.$$

Deferring the technical details to Lemma 3.3.2, we momentarily assume that $\text{dist}(F, G) > 0$, which allows us to write

$$I(F, G) = 2 \int_F \int_G \partial_1^2 \Gamma(x - y) dx dy.$$

Note that $\partial_1^2 \Gamma(z)$ takes both signs and vanishes on the double cone generated by $|z|^2 = nz_1^2$. If the distance between F and G is large compared to their diameters, I is approximated by the well-known formula for the interaction energy of two dipoles with dipole moments $|F|e_1$ and $|G|e_1$ respectively (see (3.26) for the precise formulation).

Our analysis is not restricted to global minimizers. In fact, most of our results extend also to local minimizers with respect to the metric $d(F, G)^1 := |F \Delta G|$,

¹More precisely, d is a pseudometric on $\{F \subset \mathbb{R}^n : |F| < \infty\}$ and becomes a metric upon identifying sets that agree up to a set of measure zero.

where Δ denotes the symmetric difference of sets. Furthermore, we say that a sequence of sets $(F_k)_{k \in \mathbb{N}}$ *converges locally* to F if

$$|(F_k \Delta F) \cap K| \rightarrow 0 \quad \text{for all compact } K \subset \mathbb{R}^n. \quad (3.5)$$

Although our focus is on dimensions $n = 2$ and $n = 3$, we will allow arbitrary $n \geq 2$ when this does not require additional work.

We introduce more notation and recall results that we use frequently throughout this chapter. Let $F \subset \mathbb{R}^n$ be a Lebesgue measurable set and let $A \subset \mathbb{R}^n$ be open. The relative perimeter of F in A is defined by

$$\mathcal{P}(F; A) = \sup \left\{ \int_F \operatorname{div} T(x) \, dx : T \in C_c^1(A; \mathbb{R}^n) \text{ and } \sup |T| \leq 1 \right\}.$$

Note that $\mathcal{P}(\cdot; A)$ is lower semi continuous with respect to local convergence of sets (see, e.g., [67, p.126]). We say that F has finite perimeter, if $\mathcal{P}(F) := \mathcal{P}(F; \mathbb{R}^n) < \infty$ which we assume for the rest of this paragraph. Then the distributional derivative of the characteristic function χ_F of F can be represented as the integration against an \mathbb{R}^n -valued Radon measure $D\chi_F$. Its support

$$\operatorname{supp} D\chi_F = \{x \in \mathbb{R}^n : 0 < |B_r(x) \cap F| < |B_r| \text{ for all } r > 0\} \quad (3.6)$$

serves as a measure theoretic notion of boundary. In contrast to the topological boundary ∂F of F , $\operatorname{supp} D\chi_F$ is well-defined for \mathcal{H}^n equivalence classes of sets. For $x \in \operatorname{supp} D\chi_F$ we define the measure theoretic outer normal $\nu(x)$ whenever

$$\nu(x) = - \lim_{r \rightarrow 0^+} \frac{D\chi_F(B_r(x))}{\mathcal{P}(F; (B_r(x)))} \text{ exists and belongs to } \mathbb{S}^{n-1}. \quad (3.7)$$

The set of points $x \in \operatorname{supp} D\chi_F$ where $\nu(x)$ is defined is called the reduced boundary $\partial^* F$ of F . Note that $\partial^* F \subset \operatorname{supp} D\chi_F \subset \partial F$. Moreover, for every F , there is a Borel set \tilde{F} which is \mathcal{H}^n -equivalent to F and has minimal topological boundary $\operatorname{supp} D\chi_{\tilde{F}} = \partial \tilde{F}$ (see e.g. [67, Proposition 12.19]). The essential interior \mathring{F}^M of F is the set of all points with density one

$$\mathring{F}^M = \left\{ x \in \mathbb{R}^n : \liminf_{r \rightarrow 0} \frac{|F \cap B_r(x)|}{|B_r|} = 1 \right\}. \quad (3.8)$$

It satisfies $|\mathring{F}^M \Delta F| = 0$. For our analysis, \mathring{F}^M is a particularly useful representative of the \mathcal{H}^n equivalence class of F (see section 3.2).

By De Giorgi's structure theorem, we have $D\chi_F = -\nu \, d\mathcal{H}^{n-1} \llcorner \partial^* F$ and in particular (see e.g. [67, p.170])

$$\int_F \operatorname{div} T \, dx = \int_{\partial^* F} T \cdot \nu \, d\mathcal{H}^{n-1} \quad \text{for all } T \in C_c^1(\mathbb{R}^n; \mathbb{R}^n). \quad (3.9)$$

For a vector field $T \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ the boundary divergence $\operatorname{div}_F T : \partial^* F \rightarrow \mathbb{R}$ of T on Ω is given by

$$\operatorname{div}_F T = \operatorname{div} T - \nu \cdot \nabla T \nu. \quad (3.10)$$

Moreover, we recall Newton's kernel

$$\Gamma(x) = \begin{cases} \frac{1}{2\pi} \log |x| & \text{for } n = 2, \\ \frac{1}{n(2-n)\omega_n} \frac{1}{|x|^{n-2}} & \text{for } n \geq 3, \end{cases} \quad (3.11)$$

where ω_n denotes the volume of the unit ball in \mathbb{R}^n and the sign of Γ is chosen such that $\Delta \Gamma = \delta$ distributionally.

3.1 A simple model for uniaxial magnetic domains

Our analysis of (3.1) is motivated by questions related to the nucleation of magnetic domains in thin ferromagnetic films exposed to an external magnetic field close to saturation. In the following, we explain heuristically why local minimizers of (3.1) are a suitable simple model for a single magnetic domain. However, the goal of this chapter is to study (3.1) and we do not claim that there is a rigorous connection between (3.1) and the full micromagnetic energy (0.1).

Consider a film $(0, t) \times \mathbb{R}^2$ of a ferromagnetic material with non-dimensionalized thickness t and artificial period $\ell \gg t$ in the plane. Assume furthermore that

- (a) the magnetization tends to be aligned perpendicular to the film plane on most of the sample, i.e. $m \approx \pm e_1$ and
- (b) the width of domain walls is small compared to the typical length scale of domains.

Under the above assumptions, the micromagnetic energy (0.1) should be well approximated by the following sharp interface model for the out of plane component $m_1 \in BV((0, t) \times \mathbb{T}_\ell^2, \{-1, 1\})$,

$$\tilde{E}(m_1) = 2 \int_{(0, t) \times \mathbb{T}_\ell^2} l_{\text{ex}} \sqrt{Q} |\nabla m_1| - h_{\text{ext}} m_1 \, dx + \int_{\mathbb{R} \times \mathbb{T}_\ell^2} |h|^2 \, dx. \quad (3.12)$$

In fact, this has been proved rigorously in [81] for the regime $Q \gg 1$, $l_{\text{ex}} Q^{\frac{1}{2}} \ll 1$ and $(l_{\text{ex}} Q^{\frac{1}{2}})^{\frac{1}{3}} t^{\frac{2}{3}} \ll \ell$ in the absence of an external field. The case with applied field is discussed in [56]. The last term in (3.12) denotes the stray field energy. Note that it can also be written as in (3.1) in the form

$$\int_{\mathbb{R} \times \mathbb{T}_\ell^2} |h|^2 \, dx = \int_{\mathbb{R} \times [0, \ell)^2} |\nabla \Phi|^2 \, dx$$

where $\nabla \Phi_\Omega \in L^2(\mathbb{R} \times \mathbb{T}_\ell^2; \mathbb{R}^3)$ solves $\text{div}(\nabla \Phi) = \partial_1 m_1$. We identify m_1 with $\Omega \subset \mathbb{R}^3$ via $m_1 = -1 + 2\chi_\Omega$ and consider (3.1) as a full space analog to (3.12). The Zeeman energy in (3.12), which determines the volume fraction of the two phases, has been replaced by the volume constraint (3.3).

In addition to (a) and (b), the interpretation of local minimizers of (3.1) as single magnetic domains is limited by the implicit assumption that

- (c) the single domain is sufficiently far away from other domains and the sample boundary.

However, we hope that our analysis for this toy problem will be useful for further analysis of the full micromagnetic energy.

3.2 Main results

In this section we state our main results. The first theorem asserts the existence of minimizers for all prescribed volumes $V \geq 0$.

Theorem 3.2.1 (Existence). *For every $V \geq 0$ there exists an $\Omega \in \mathcal{C}_V$ with*

$$\mathcal{E}(\Omega) = \inf_{F \in \mathcal{C}_V} \mathcal{E}(F).$$

The proof is based on arguments in the spirit of the concentration compactness principle [65]. A simple but fruitful observation is that the minimal energy

$$e(V) := \inf_{F \in \mathcal{C}_V} \mathcal{E}(F) \quad (3.13)$$

is strictly subadditive (see Lemma 3.4.1). This information is used to rule out partial vanishing of volume for the limit of a minimal sequence. Since the strict subadditivity mainly relies on the scaling identities

$$\mathcal{P}(\mu\Omega) = \mu^{n-1}\mathcal{P}(\Omega) \quad \text{and} \quad \mathcal{N}(\mu\Omega) = \mu^n\mathcal{N}(\Omega) \quad \text{for all } \mu > 0,$$

we want to point out that the argument also applies to related models, e.g. for elastic inclusions as in [54].

We turn to the regularity of local minimizers. Recall that \mathcal{E} is oblivious to changes on \mathcal{H}^n negligible sets. Hence we focus on a suitable representative from each \mathcal{H}^n equivalence class of minimizers.

Definition 3.2.2 (Regular local minimizer). The set $\Omega \in \mathcal{C}_V$ is called a *regular local minimizer* of \mathcal{E} if the following holds.

- (i) There is $\delta > 0$ such that $\mathcal{E}(\Omega) \leq \mathcal{E}(F)$ for all $F \in \mathcal{C}_V$ with $|F \Delta \Omega| < \delta$.
- (ii) Ω equals its essential interior $\left\{x \in \mathbb{R}^n : \liminf_{r \rightarrow 0} \frac{|\Omega \cap B_r(x)|}{|B_r|} = 1\right\}$.

Our main regularity result is the following.

Theorem 3.2.3 (Regularity of the boundary). *Let $2 \leq n \leq 7$ and let $\Omega \in \mathcal{C}_V$ be a regular local minimizer of \mathcal{E} (see Def. 3.2.2). Then Ω is an open bounded set with smooth boundary.*

The proof of Theorem 3.2.3 uses the regularity theory for quasi-minimizers of the perimeter [25, 91]. The assumption $n \leq 7$ is convenient because it prevents the existence of singular points. However, we do not know whether the restriction to $n \leq 7$ is essential for Theorem 3.2.3.

We continue with the regularity of the stray field $-\nabla\Phi_\Omega$ associated to a local minimizer Ω of \mathcal{E} . Note that for a cube $Q = (0, 1)^n$, some components of $\nabla\Phi_Q$ exhibit logarithmic singularities at the edges and corners of Q (see e.g. [92]). In contrast, the smooth boundary of a regular local minimizer admits to use classical results for the so-called single layer potential [39, 72] (see also Theorem 3.3.3). In particular, for an open bounded set Ω with $C^{1,\alpha}$ boundary ($\alpha \in (0, 1)$), we introduce the quantity $\langle \nabla\Phi_\Omega \rangle : \partial\Omega \rightarrow \mathbb{R}^n$ given by

$$\langle \nabla\Phi_\Omega \rangle(x) = -\lim_{\delta \rightarrow 0} \int_{\partial\Omega \setminus B_\delta(x)} \nabla\Gamma(x-y)\nu_1(y) \, d\mathcal{H}^{n-1}(y) \quad \text{for } x \in \partial\Omega, \quad (3.14)$$

where $\nu : \partial\Omega \rightarrow \mathbb{S}^{n-1}$ denotes the outward pointing normal vector to $\partial\Omega$. We obtain the following regularity result for the stray field $-\nabla\Phi_\Omega$ of a regular local minimizer Ω of \mathcal{E} .

Corollary 3.2.4 (Regularity of the stray field). *Let $2 \leq n \leq 7$ and let $\Omega \in \mathcal{C}_V$ be a regular local minimizer of \mathcal{E} (see Def. 3.2.2). Then $\nabla\Phi_\Omega$ (see (3.2)) satisfies $\nabla\Phi_\Omega \in L^\infty(\mathbb{R}^n; \mathbb{R}^n)$. Moreover, $\nabla\Phi_\Omega$ is harmonic on $\mathbb{R}^n \setminus \partial\Omega$ and has smooth extensions to $\partial\Omega$ from Ω and $\mathbb{R}^n \setminus \overline{\Omega}$, denoted by $(\nabla\Phi_\Omega)_i$ and $(\nabla\Phi_\Omega)_e$ respectively. They satisfy the jump relations*

$$\begin{cases} (\nabla\Phi_\Omega)_i = \langle \nabla\Phi_\Omega \rangle - \frac{\nu_1}{2}\nu, \\ (\nabla\Phi_\Omega)_e = \langle \nabla\Phi_\Omega \rangle + \frac{\nu_1}{2}\nu, \end{cases} \quad \text{on } \partial\Omega,$$

where $\nu \in C^\infty(\partial\Omega; \mathbb{S}^{n-1})$ denotes the outward pointing normal vector.

For a regular local minimizer, the value of its stray field on the boundary is related to its local geometry. Exploiting stationarity of a local minimizers Ω of \mathcal{E} with respect to inner variations, we obtain the following optimality condition.

Theorem 3.2.5 (Noether equation). *Let $2 \leq n \leq 7$ and let $\Omega \in \mathcal{C}_V$ be a regular local minimizer of \mathcal{E} (see Def. 3.2.2). Let H_Ω denote the sum of the principal curvatures of $\partial\Omega$ and let $\langle \partial_1\Phi_\Omega \rangle$ be given by (3.14). Then there is $\Lambda \in \mathbb{R}$ such that*

$$H_\Omega + 2\langle \partial_1\Phi_\Omega \rangle = \Lambda \quad \text{on } \partial\Omega. \quad (3.15)$$

Following [36, p.146] we call (3.15) the Noether equation associated to \mathcal{E} because it originates from taking inner variations. However, we note that such equations are sometimes also called Euler-Lagrange equations.

The proof of Theorems 3.2.3 and 3.2.5 are strongly related. First, we invoke the regularity theory for quasi-minimizers of the perimeter [25, 91] to deduce $C^{1,\alpha}$ -regularity of the boundary. We then exploit stationarity with respect to inner variations, incorporating the volume constraint with the aid of a Lagrange multiplier. This shows that regular local minimizers satisfy (3.15) in a weak form (see (3.16) below) and are hence regular critical points in the sense of the following definition.

Definition 3.2.6 (Regular critical point). $\Omega \in \mathcal{C}_V$ is called a regular critical point, if

- (i) Ω is open and bounded with $C^{1,\alpha}$ boundary for some $\alpha \in (0, 1)$ and
- (ii) there is $\Lambda \in \mathbb{R}$ such that

$$\int_{\partial\Omega} \operatorname{div}_\Omega T \, d\mathcal{H}^{n-1} + \int_{\partial\Omega} (2\langle \partial_1 \Phi_\Omega \rangle - \Lambda) (T \cdot \nu) \, d\mathcal{H}^{n-1} = 0 \quad (3.16)$$

holds for all $T \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$ where $\operatorname{div}_\Omega T = \operatorname{div} T - \nu \cdot \nabla T \nu$ denotes the boundary divergence of T .

Finally, we use a bootstrap argument for (3.16) to show that regular critical points have smooth boundary and satisfy (3.15).

We turn to topological properties of local minimizers. In particular, we show that regular local minimizers are connected.

Theorem 3.2.7 (Connectedness). *Let $2 \leq n \leq 7$ and let $\Omega \in \mathcal{C}_V$ be a regular local minimizer of \mathcal{E} (see Def. 3.2.2). Then Ω , $\mathbb{R}^n \setminus \overline{\Omega}$ and $\partial\Omega$ are connected. For $n = 2$, Ω is homeomorphic to a ball.*

For $n \geq 3$, however, the situation is more complex. While Theorem 3.2.7 rules out the existence of “cavities” as in $\Omega = B_1 \setminus \overline{B}_{\frac{1}{2}}$, it does not exclude the possibility that $\partial\Omega$ is a k -fold torus. In particular, we do not know whether regular local minimizers are simply connected.

To give a glimpse at our proof that regular minimizers are connected, we argue by contradiction and consider a disconnected global minimizer $A \cup B$ with A, B open, nonempty disjoint. A key observation is that the energy of partially shifted configurations $y \mapsto \mathcal{E}(A \cup (y + B))$ satisfies a strong maximum principle and hence must be constant (on the connected component of 0 in $\{y \in \mathbb{R}^n : \operatorname{dist}(A, y + B) > 0\}$). This allows to construct another minimizer which violates Theorem 3.2.3 and thus yields the desired contradiction.

To further characterize low energy configurations for \mathcal{E} , we study the scaling of the minimal energy for $n = 3$.

Theorem 3.2.8 (Scaling of the minimal energy). *Let $n = 3$ and define $f : (0, \infty) \rightarrow \mathbb{R}$ by*

$$f(V) := \begin{cases} V^{\frac{2}{3}} & \text{for } V \leq 1, \\ V^{\frac{5}{7}}(\log eV)^{\frac{1}{7}} & \text{for } V \geq 1. \end{cases}$$

Then there are universal constants $c, C > 0$ such that

$$cf(V) \leq \min_{F \in \mathcal{C}_V} \mathcal{E}(F) \leq Cf(V).$$

The optimal scaling of the energy for large V is achieved by prolate ellipsoids of length $L \sim V^{\frac{3}{7}}(\log V)^{\frac{2}{7}}$ in the e_1 -direction and radius $R \sim V^{\frac{2}{7}}(\log V)^{-\frac{1}{7}}$ in the plane perpendicular to e_1 (which become slender for large V in the sense that $R/L \rightarrow 0$ for $V \rightarrow \infty$). The ansatz free lower bound is based on a geometric construction from [17] which has also been used in the micromagnetic setting in [56].

Organization: The remainder of chapter 3 is organized as follows: In section 3.3 we establish elementary properties of the potential and review results from potential theory that we will use throughout this work. Existence of minimizers is proved in section 3.4. The proof of the regularity for local minimizers takes up sections 3.5 and 3.6: More precisely, initial $C^{1,\alpha}$ -regularity of the boundary of a local minimizer is shown in section 3.5. In turn, the first inner variation of \mathcal{E} at $C^{1,\alpha}$ -sets is computed in subsection 3.6.1. Finally, higher regularity for regular critical points and of the associated stray field is proved in subsection 3.6.2. Connectedness of local minimizers is proved in section 3.7 and the scaling of the minimal energy is proved in section 3.8.

3.3 Preliminaries

We record several basic properties of Φ_Ω and \mathcal{N} for future use.

Lemma 3.3.1 (Properties of Φ_Ω). *Let $\Omega \subset \mathbb{R}^n$ satisfy $|\Omega| < \infty$. Then the following holds*

(i) *Problem (3.2) has a unique distributional solution, given by*

$$\Phi_\Omega(x) = \int_{\mathbb{R}^n} \partial_1 \Gamma(x - y) \chi_\Omega(y) \, dy \quad \text{for a.e. } x \in \mathbb{R}^n,$$

where Γ is Newton's kernel.

(ii) The gradient $\nabla\Phi_\Omega$ admits the following representation using a Calderón-Zygmund kernel

$$\partial_k\Phi_\Omega(x) = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \partial_{k,1}^2 \Gamma(x-y) \chi_\Omega(y) \, dy + \frac{\delta_{i,k}}{n} \chi_\Omega(x) \quad (3.17)$$

for every $1 \leq k \leq n$ and almost every $x \in \mathbb{R}^n$. Here $\delta_{i,k}$ denotes the Kronecker Delta with $\delta_{i,k} = 1$ if $i = k$ and $\delta_{i,k} = 0$ otherwise.

Moreover, for all $p \in (1, \infty)$, we have $\Phi_\Omega \in W_{loc}^{1,p}(\mathbb{R}^n)$ and there is a constant $C_{n,p}$ such that

$$\|\nabla\Phi_\Omega\|_{L^p(\mathbb{R}^n)} \leq C_{n,p} |\Omega|^{\frac{1}{p}}. \quad (3.18)$$

(iii) If Ω is a bounded set of finite perimeter and $\mathcal{H}^n(\partial\Omega) = 0$, then Φ_Ω has the alternative representation

$$\Phi_\Omega(x) = - \int_{\partial^*\Omega} \Gamma(x-y) \nu_1(y) \, d\mathcal{H}^{n-1}(y)$$

for almost every $x \in \mathbb{R}^n$.

Lemma 3.3.1 states in particular that the solution $\partial_1\Gamma * \chi_\Omega$ coincides with the solution obtained by L^p -theory (upon fixing a constant). Of course, this is easily verified if χ_Ω is replaced by some $f \in C_c^\infty(\mathbb{R}^n)$ and extends to χ_Ω by an approximation argument. However, since we have not been able to find a reference which applies precisely to our setting, we give a few details below.

Proof. The proof of item (i) is standard and provided in Lemma B.1 in the appendix for the sake of completeness of the presentation.

We turn to the proof of (ii). Let $T : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n; \mathbb{R}^n)$ be given by

$$(Tf)_k(x) = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \partial_k \partial_1 \Gamma(x-y) f(y) \, dy + \frac{\delta_{1,k}}{n} f(x) \quad (3.19)$$

for all $1 \leq k \leq n$. Then T is a linear and bounded operator for any $p \in (1, \infty)$ and the convergence as $\varepsilon \rightarrow 0$ in (3.19) holds in $L^p(\mathbb{R}^n; \mathbb{R}^n)$ (see, e.g., [89, II.4.2 Theorem 3, p.39] and note that $\int_{\mathbb{S}^{n-1}} \partial_j \partial_k \Gamma \, d\mathcal{H}^{n-1} = 0$). A direct calculation shows that (see Lemma B.2)

$$\nabla(\partial_1\Gamma * f) = Tf \quad \text{for all } f \in C_c^\infty(\mathbb{R}^n). \quad (3.20)$$

This identity extends to the case $f = \chi_\Omega$ by the following approximation argument. Let $f_i \in C_c^\infty(\mathbb{R}^n)$ such that $f_i \rightarrow \chi_\Omega$ in $L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ and define $\Psi_i = \partial_1 \Gamma * f_i$. By (3.20) and continuity of T , we have

$$\nabla \Psi_i = T f_i \rightarrow T \chi_\Omega \quad \text{in } L^2(\mathbb{R}^n). \quad (3.21)$$

It is now sufficient to show $\nabla \Psi_i \rightharpoonup \nabla \Psi_\Omega$ (weakly) in $L^p(\mathbb{R}^n; \mathbb{R}^n)$, then (3.17) and (3.18) follow from (3.21) and the L^p -estimates for T . We first show that we have $\Psi_i \rightarrow \Phi_\Omega$ in $L^1_{loc}(\mathbb{R}^n)$. Indeed, since $|\partial_1 \Gamma| \leq |\partial_1 \Gamma| \chi_{B_1} + \chi_{\mathbb{R}^n \setminus B_1}$ we may split the integral and with the aid of Fubini's Theorem, we get for any $R > 0$

$$\begin{aligned} \int_{B_R} |\Phi_\Omega(x) - \Psi_i(x)| dx &\leq \int_{B_R} \int_{B_1} |\partial_1 \Gamma(y)| |\chi_\Omega(x-y) - f_i(x-y)| dy dx \\ &\quad + \int_{B_R} \int_{\mathbb{R}^n \setminus B_1} |\chi_\Omega(x-y) - f_i(x-y)| dy dx \\ &\leq \left(\int_{B_1} |\partial_1 \Gamma(y)| dy + |B_R| \right) \|\chi_\Omega - f_i\|_{L^1(\mathbb{R}^n)} \rightarrow 0. \end{aligned}$$

Let $R > 0$ and define $(\Psi_i)_R := \int_{B_R} \Psi_i dx$. Since $\limsup_{i \rightarrow \infty} \|\nabla \Psi_i\|_{L^p(\mathbb{R}^n; \mathbb{R}^n)} < \infty$ by (3.21), Poincaré's inequality implies that the sequence $\Psi_i - (\Psi_i)_R$ is bounded in $W^{1,p}(B_R)$. By weak compactness, there is a subsequence (not relabeled) and some $g \in W^{1,p}(B_R)$ such that

$$\nabla \Psi_i \rightharpoonup \nabla g \quad (\text{weakly}) \text{ in } L^p(B_R; \mathbb{R}^n) \text{ and } \Psi_i - (\Psi_i)_R \rightarrow g \text{ in } L^p(B_R).$$

Uniqueness of the limit and $\Psi_i \rightarrow \Phi_\Omega$ in $L^1_{loc}(\mathbb{R}^n)$ imply $\nabla \Psi_i \rightharpoonup \nabla \Phi_\Omega$ in $L^p(B_R; \mathbb{R}^n)$. Since $R > 0$ was arbitrary, we get

$$\nabla \Psi_i \rightharpoonup \nabla \Phi_\Omega \text{ in } L^p(\mathbb{R}^n),$$

which, together with (3.21), implies $\nabla \Psi_\Omega = T \chi_\Omega$. Since $\chi_\Omega \in L^p(\mathbb{R}^n)$ for every $p \in (1, \infty)$, boundedness of T yields a constant $C_{n,p}$ such that

$$\|\nabla \Psi_\Omega\|_{L^p} \leq C_{n,p} \|\chi_\Omega\|_{L^p} = C_{n,p} |\Omega|^{\frac{1}{p}}.$$

Turning to the proof of item (iii), we introduce $\psi_\Omega : \mathbb{R}^n \rightarrow \mathbb{R}$, given by

$$\psi_\Omega(x) = - \int_{\partial^* \Omega} \Gamma(x-y) \nu_1(y) d\mathcal{H}^{n-1}(y).$$

The function ψ_Ω is well defined for $x \notin \partial\Omega$ because then $\Gamma(x - \cdot)$ is continuous on the compact set $\partial\Omega$. Let $\rho \in C_c^\infty(B_1)$ with $\int_{\mathbb{R}^n} \rho \, dx = 1$ and set $\rho_\varepsilon = \frac{1}{\varepsilon^n} \rho\left(\frac{\cdot}{\varepsilon}\right)$ for all $\varepsilon > 0$. Applying the weak Gauss-Green formula (3.9) for $\Gamma * \rho_\varepsilon$ yields

$$-\int_{\partial^*\Omega} (\Gamma * \rho_\varepsilon)(x - y) \nu_1(y) \, d\mathcal{H}^{n-1}(y) = \int_{\mathbb{R}^n} (\partial_1 \Gamma * \rho_\varepsilon)(x - y) \chi_\Omega(y) \, dy.$$

In the limit $\varepsilon \rightarrow 0$, this implies $\psi_\Omega(x) = \Phi_\Omega(x)$ for all $x \notin \partial\Omega$ and hence almost everywhere. \square

We continue by recording several basic estimates for the nonlocal term.

Lemma 3.3.2 (Identities and estimates and for the nonlocal term). *Let $\Omega, F, G \subset \mathbb{R}^n$ with finite measure. Then the following holds*

(i) $\mathcal{N}(\Omega)$ has the Fourier representation

$$\mathcal{N}(\Omega) = \int_{\mathbb{R}^n} \frac{\xi_1^2}{|\xi|^2} |\widehat{\chi}_\Omega(\xi)|^2 \, d\xi. \quad (3.22)$$

(ii) Let $p \in (1, \infty)$ and $q := \frac{p}{p-1}$. Then there is a constant $C_{n,p}$ such that

$$|\mathcal{N}(F) - \mathcal{N}(G)| \leq C_{n,p} |F \cup G|^{\frac{1}{p}} |F \Delta G|^{\frac{1}{q}}. \quad (3.23)$$

Moreover, the interaction energy $I(F, G)$ (see (3.4)) satisfies

$$I(F, G) \leq C_{n,p} |F|^{\frac{1}{p}} |G|^{\frac{1}{q}}. \quad (3.24)$$

(iii) For $J \in \{F, G\}$, define the center of mass x_J and the dipole moment μ_J by

$$x_J = \int_J x \, dx \quad \text{and} \quad \mu_J = |J| e_1 \quad (3.25)$$

and abbreviate $r = x_F - x_G$. If F and G have positive distance the interaction energy $I(F, G)$ has the asymptotic form

$$\begin{aligned} I(F, G) = & \frac{2}{n\omega_n} \left(\frac{\mu_F \cdot \mu_G |r|^2 - n(\mu_F \cdot r)(\mu_G \cdot r)}{|r|^{n+2}} \right) \\ & + O\left(|F||G| \frac{\text{diam}(F) + \text{diam}(G)}{r^{n+1}}\right). \end{aligned} \quad (3.26)$$

Proof. To show (3.22), we approximate χ_Ω by a sequence of smooth functions $f_k \in C_c^\infty(\mathbb{R}^n)$ such that $f_k \rightarrow \chi_\Omega$ in $L^2(\mathbb{R}^n)$. We introduce $\Phi_k \in H^2(\mathbb{R}^n)$ as the unique solution of $\Delta \Phi_k = \partial_1 f_k$. Since $\widehat{\Phi}_k(\xi) = -\frac{i\xi_1}{|\xi|^2} \widehat{f}_k(\xi)$ and $\nabla \Phi_k \rightarrow \nabla \Phi_\Omega$ in $L^2(\mathbb{R}^n)$, we get (3.22)

$$\begin{aligned} \mathcal{N}(\Omega) &= \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} |\nabla \Phi_k|^2 \, dx = \int_{\mathbb{R}^n} |\xi|^2 \left| \widehat{\Phi}_k(\xi) \right|^2 \, d\xi \\ &= \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} \frac{\xi_1^2}{|\xi|^2} |\widehat{f}_k(\xi)|^2 \, d\xi = \int_{\mathbb{R}^3} \frac{\xi_1^2}{|\xi|^2} |\widehat{\chi_\Omega}(\xi)|^2 \, d\xi. \end{aligned}$$

Estimates (3.23) and (3.24) are direct consequences of Hölder's inequality and the elliptic L^p -estimates (3.18). Indeed, for all $p \in (1, \infty)$ we have

$$\begin{aligned} |\mathcal{N}(F) - \mathcal{N}(G)| &\leq \int_{\mathbb{R}^n} \left| |\nabla \Phi_F|^2 - |\nabla \Phi_G|^2 \right| \, dx \\ &= \int_{\mathbb{R}^n} |\nabla \Phi_F + \nabla \Phi_G| |\nabla \Phi_F - \nabla \Phi_G| \, dx \\ &\leq C_{n,p} \|\chi_F + \chi_G\|_{L^p} \|\chi_F - \chi_G\|_{L^q} \\ &\leq C_{n,p} |F \cup G|^{\frac{1}{p}} |F \Delta G|^{\frac{1}{q}}. \end{aligned}$$

Similarly, we obtain for the interaction energy

$$\begin{aligned} I(F, G) &= 2 \int_{\mathbb{R}^n} \nabla \Phi_F \cdot \nabla \Phi_G \, dx \leq 2 \|\nabla \Phi_F\|_{L^p} \|\nabla \Phi_G\|_{L^q} \\ &\leq C_{n,p} \|\chi_F\|_{L^p} \|\chi_G\|_{L^q} \leq C_{n,p} |F|^{\frac{1}{p}} |G|^{\frac{1}{q}}. \end{aligned}$$

To prove (3.26), we use the weak formulation of (3.2) to obtain

$$I(F, G) = 2 \int_G \partial_1 \Phi_F \, dx.$$

Inserting (3.17) into the previous expression, we get

$$I(F, G) = 2 \int_G \left(\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \partial_1^2 \Gamma(x-y) \chi_F(y) \, dy \right) \, dx + \frac{2}{n} |F \cap G|.$$

Since $\text{dist}(F, G) > 0$ the above expression simplifies to

$$I(F, G) = 2 \int_G \int_F \partial_1^2 \Gamma(x-y) \, dy \, dx, \quad (3.27)$$

where the strongly singular kernel $\partial_1^2 \Gamma(z)$ is given by

$$\partial_1^2 \Gamma(z) = \frac{1}{n\omega_n} \frac{|z|^2 - nz_1^2}{|z|^{n+2}}. \quad (3.28)$$

A short calculation shows that (3.28) satisfies

$$|\partial_1^2 \Gamma(x - y) - \partial_1^2 \Gamma(x_F - x_G)| \lesssim \frac{\text{diam}(F) + \text{diam}(G)}{|x_F - x_G|^{n+1}} \quad (3.29)$$

for all $x \in F, y \in G$. Inserting (3.29) and (3.25) into (3.27) yields the claim (3.26). \square

We will use the following classical results of potential theory.

Theorem 3.3.3 (Fine properties of the potential). *Let $k \geq 0$ be an integer and $\alpha \in (0, 1)$. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with $C^{k+1, \alpha}$ boundary and let $f \in C^{k, \alpha}(\partial\Omega)$. Define the potential $\Psi : \mathbb{R}^n \rightarrow \mathbb{R}$ by*

$$\Psi(x) = \int_{\partial\Omega} \Gamma(x - y) f(y) d\mathcal{H}^{n-1}(y).$$

Furthermore, we define the direct value of the gradient $\langle \nabla \Psi \rangle : \partial\Omega \rightarrow \mathbb{R}^n$ by

$$\langle \nabla \Psi \rangle(x) = \lim_{\varepsilon \rightarrow 0} \int_{\partial\Omega \setminus B_\varepsilon(x)} \nabla \Gamma(x - y) f(y) d\mathcal{H}^{n-1}(y) \quad \text{for all } x \in \partial\Omega.$$

Then the following holds:

- (i) *The potential Ψ is continuous on \mathbb{R}^n and harmonic on $\mathbb{R}^n \setminus \partial\Omega$. Moreover, the restriction of Ψ to Ω has an extension $\Psi_i \in C^{1+k, \alpha}(\overline{\Omega})$ and, likewise, the restriction of Ψ to $\mathbb{R}^n \setminus \overline{\Omega}$ has an extension $\Psi_e \in C^{1+k, \alpha}(\overline{\Omega})$.*
- (ii) *The extensions $\Psi_{i/e}$ from (i) satisfy the jump relations*

$$\begin{aligned} \nabla \Psi_i &= +\frac{f}{2} \nu + \langle \nabla \Psi \rangle && \text{on } \partial\Omega, \\ \nabla \Psi_e &= -\frac{f}{2} \nu + \langle \nabla \Psi \rangle && \text{on } \partial\Omega, \end{aligned}$$

where $\nu : \partial\Omega \rightarrow \mathbb{R}^n$ denotes the outward pointing unit normal.

- (iii) *In particular, $\langle \nabla \Psi \rangle \in C^{k, \alpha}(\partial\Omega; \mathbb{R}^n)$.*

The case $k = 0$ was proved by Giraud, see in particular [39, chapter 7] (in French). The proof for general k is due to Miranda [72, Th. 2.I] (in Italian). See also [73, Theorems 14.V and 14.VII] and [71, p.367] for related but weaker statements in English. The jump relations continue to hold even for Lipschitz boundaries and $f \in L^p(\partial\Omega)$ in an almost everywhere sense for so-called nontangential limits, see, e.g., [93, Thm. 1.11]).

We want to point out that Theorem 3.2.3 only depends on item (iii) in Theorem 3.3.3 above. Items (i) and (ii) are only used to obtain Corollary 3.2.4.

3.4 Existence of minimizers

In this section we prove Theorem 3.2.1 via the direct method in the calculus of variations and arguments in the spirit of the concentration compactness principle [65]. Since the main difficulty is to exclude vanishing of volume (i.e. “mass”) in the limit, this approach (in the context of geometric variational problems) has been called “method of the vanishing mass” by Frank and Lieb (see [33] for a beautiful presentation of this method). We want to emphasize that the proof mainly uses the sublinear scaling of the energy together with mild decay properties of the nonlocal interaction and is oblivious to the specific structure of the nonlocal term.

We begin with the proof of the subadditivity of the minimal energy.

Lemma 3.4.1. *The minimal energy e (see (3.13)) is continuous and strictly subadditive*

$$e(V_1) + e(V_2) > e(V_1 + V_2) \quad \text{for all } V_1, V_2 > 0. \quad (3.30)$$

Proof. Using the scaling properties of the energy, we obtain

$$\begin{aligned} e(V) &= \inf_{\Omega \in \mathcal{C}_V} (\mathcal{P}(\Omega) + \mathcal{N}(\Omega)) = \inf_{\hat{\Omega} \in \mathcal{C}_1} \left(\mathcal{P}(\hat{\Omega}) V^{\frac{n-1}{n}} + \mathcal{N}(\hat{\Omega}) V \right) \\ &= \inf_{\hat{\Omega} \in \mathcal{C}_1} V \left(\mathcal{P}(\hat{\Omega}) V^{-\frac{1}{n}} + \mathcal{N}(\hat{\Omega}) \right) \quad \text{for all } V > 0. \end{aligned} \quad (3.31)$$

In particular, (3.31) shows that the minimal energy is the pointwise infimum over a family of concave functions and hence concave and continuous on $(0, \infty)$. Since $e(0) = 0$, this already implies that e is subadditive. In the following, we will use the isoperimetric inequality $\inf_{\Omega \in \mathcal{C}_1} \mathcal{P}(\Omega) \geq n\omega_n$ to show that the subadditivity is actually strict. Inserting a zero, we may rewrite (3.31) as

$$\begin{aligned} e(V_i) &= \inf_{\hat{\Omega} \in \mathcal{C}_1} V_i \left(\mathcal{P}(\hat{\Omega}) (V_1 + V_2)^{-\frac{1}{n}} + \mathcal{N}(\hat{\Omega}) \right. \\ &\quad \left. + \mathcal{P}(\hat{\Omega}) \left(V_i^{-\frac{1}{n}} - (V_1 + V_2)^{-\frac{1}{n}} \right) \right) \end{aligned} \quad (3.32)$$

for $i = 1, 2$ and all $V_1, V_2 > 0$. Inserting $\mathcal{P}(\hat{\Omega}) \geq n\omega_n$ into (3.32) yields

$$\begin{aligned} e(V_i) &\geq \inf_{\hat{\Omega} \in \mathcal{C}_1} V_i \left(\mathcal{P}(\hat{\Omega}) (V_1 + V_2)^{-\frac{1}{n}} + \mathcal{N}(\hat{\Omega}) \right) \\ &\quad + n\omega_n \left(V_i^{\frac{n-1}{n}} - V_i (V_1 + V_2)^{-\frac{1}{n}} \right). \end{aligned} \quad (3.33)$$

Adding (3.33) for $i = 1, 2$ and observing that $x^\alpha + (1 - x)^\alpha > 1$ for $x \in (0, 1)$ and $\alpha \in [0, 1)$ we get

$$\begin{aligned} e(V_1) + e(V_2) &\stackrel{(3.33)}{\geq} (V_1 + V_2) \inf_{\widehat{\Omega} \in \mathcal{C}_1} \left(\mathcal{P}(\widehat{\Omega})(V_1 + V_2)^{-\frac{1}{n}} + \mathcal{N}(\widehat{\Omega}) \right) \\ &\quad + n\omega_n \left(V_1^{\frac{n-1}{n}} + V_2^{\frac{n-1}{n}} - (V_1 + V_2)^{\frac{n-1}{n}} \right) \\ &\stackrel{(3.31)}{>} e(V_1 + V_2) \quad \text{for all } V_1, V_2 > 0. \end{aligned}$$

This proves (3.30) and the proof is complete. \square

Remark 3.4.2. Since the proof of Lemma 3.4.1 only uses the scaling properties of the energy and a positive lower bound for \mathcal{P} , Lemma 3.4.1 also holds for other nonnegative terms \mathcal{N} which satisfy a scaling law of the form

$$\mathcal{N}(\lambda\Omega) = \mathcal{N}(\Omega)\lambda^{\alpha n} \quad \text{for all } \Omega \in \mathcal{C}_1 \text{ and all } \lambda > 0$$

for some $\alpha \in [0, 1]$.

Before we begin with the proof of Theorem 3.2.1, we record a compactness result.

Lemma 3.4.3 (Compactness). *Let $(\Omega_k)_{k \in \mathbb{N}}$ be a sequence with $\Omega_k \subset \mathbb{R}^n$ and*

$$\liminf_{k \rightarrow \infty} |\Omega_k| > 0 \quad \text{and} \quad \limsup_{k \rightarrow \infty} \mathcal{P}(\Omega_k) < \infty.$$

Then there exists a subsequence (still denoted by $(\Omega_k)_{k \in \mathbb{N}}$), a sequence $(a_k)_{k \in \mathbb{N}}$ of points in \mathbb{R}^n and a set $\Omega \subset \mathbb{R}^n$ with $|\Omega| > 0$ such that

$$\Omega_k - a_k \rightarrow \Omega \text{ locally for } k \rightarrow \infty.$$

A proof may be found, for instance, in [67, Cor. 12.27 and Le. 29.10]. See also [33, Prop. 2.1] for a short argument showing that it is possible to find a limit Ω with nonzero measure.

Proof of Theorem 3.2.1. We use the direct method of the calculus of variations and briefly note that \mathcal{E} is lower semi-continuous with respect to the metric $d(F, G) = |F \Delta G|$. Indeed, the lower semi-continuity of the perimeter is a classical result (see, e.g., [67, p.126]) and continuity of \mathcal{N} follows from (3.23). Let $(\Omega_k)_{k \in \mathbb{N}}$ be a minimal sequence for \mathcal{E} in \mathcal{C}_V . By a density argument we can

assume without loss of generality that Ω_k has smooth boundary. A comparison with a ball $B \in \mathcal{C}_V$ yields the uniform perimeter bound

$$\limsup_{k \rightarrow \infty} \mathcal{P}(\Omega_k) \leq \limsup_{k \rightarrow \infty} \mathcal{E}(\Omega_k) \leq \mathcal{E}(B) < \infty.$$

Now Lemma 3.4.3 asserts that, upon passing to a subsequence, we have

$$\Omega_k - a_k \rightarrow \Omega \quad \text{locally for } k \rightarrow \infty \quad (3.34)$$

for some Ω with nonzero measure. Since \mathcal{E} is invariant with respect to translations, we may assume, without loss of generality, that $a_k = 0$ for all k . Hence, it remains to show that Ω is admissible, i.e. that $|\Omega| = V$. We will show that Ω_k may be partitioned into two disjoint sets $\Omega_k = \Omega_k^{(1)} \dot{\cup} \Omega_k^{(2)}$ such that

$$\Omega_k^{(1)} \xrightarrow{d} \Omega \quad (\text{globally}) \quad (3.35)$$

and the energy is asymptotically additive with respect to this partition

$$\mathcal{E}(\Omega_k) - \left(\mathcal{E}(\Omega_k^{(1)}) + \mathcal{E}(\Omega_k^{(2)}) \right) \rightarrow 0 \quad \text{for } k \rightarrow \infty. \quad (3.36)$$

Assuming for a moment that such a partition of Ω_k exists, the proof closes as follows. By continuity of the minimal energy, we get

$$\begin{aligned} e(V) &= \lim_{k \rightarrow \infty} \mathcal{E}(\Omega_k) \stackrel{(3.36)}{=} \lim_{k \rightarrow \infty} \left(\mathcal{E}(\Omega_k^{(1)}) + \mathcal{E}(\Omega_k^{(2)}) \right) \\ &\geq \lim_{k \rightarrow \infty} \left(e(|\Omega_k^{(1)}|) + e(V - |\Omega_k^{(1)}|) \right) \stackrel{(3.35)}{=} e(|\Omega|) + e(V - |\Omega|). \end{aligned}$$

Since $0 < |\Omega| \leq V$, Lemma 3.4.1 implies $|\Omega| = V$ i.e. $\Omega \in \mathcal{C}_V$ and $\Omega_k \xrightarrow{d} \Omega$. By lower semi continuity of \mathcal{E} , the claim follows

$$\mathcal{E}(\Omega) \leq \liminf_{k \rightarrow \infty} \mathcal{E}(\Omega_k) = \min_{F \in \mathcal{C}_V} \mathcal{E}(F).$$

We will show that there exists a sequence $(r_k)_{k \in \mathbb{N}}$ such that (3.35) and (3.36) hold for $\Omega_k^{(1)} := \Omega_k \cap B_{r_k}$ and $\Omega_k^{(2)} := \Omega_k \setminus B_{r_k}$ (upon passing to a subsequence). For all $R > 0$, (3.34) implies

$$\Omega_k \cap B_R \xrightarrow{d} \Omega \cap B_R \quad \text{for } k \rightarrow \infty.$$

By a diagonal argument, we deduce that there is another subsequence (again labelled Ω_k) such that

$$\Omega_k \cap B_k \xrightarrow{d} \Omega \cap B_k \quad \text{for } k \rightarrow \infty.$$

The latter statement may be strengthened: Since

$$|\Omega \Delta (\Omega_k \cap B_r)| \leq |\Omega \setminus B_r| + |(\Omega \Delta \Omega_k) \cap B_r| \quad \text{for all } r > 0,$$

we conclude that for every sequence $(r_k)_{k \in \mathbb{N}}$ with $r_k \leq k$ and $\lim_{k \rightarrow \infty} r_k = +\infty$, we have

$$\Omega_k^{(1)} := \Omega_k \cap B_{r_k} \xrightarrow{d} \Omega, \quad (3.37)$$

$$\Omega_k^{(2)} := \Omega_k \setminus B_{r_k} \rightarrow \emptyset \quad \text{locally}, \quad (3.38)$$

for $k \rightarrow \infty$. Hence (3.35) holds. Furthermore, the coarea formula implies

$$\int_{\frac{k}{4}}^{\frac{k}{2}} \mathcal{H}^{n-1}(\Omega_k \cap \partial B_r) dr = \int_{B_{\frac{k}{2}} \setminus B_{\frac{k}{4}}} \chi_{\Omega_k} dx \leq V.$$

Thus there exists a sequence $(r_k)_{k \in \mathbb{N}}$ with $4r_k \in (k, 2k)$ such that

$$\mathcal{H}^{n-1}(\Omega_k \cap \partial B_{r_k}) \leq \frac{4V}{k}. \quad (3.39)$$

We conclude that for $\Omega_k^{(1)} = \Omega_k \cap B_{r_k}$ and $\Omega_k^{(2)} = \Omega_k \setminus B_{r_k}$, we get

$$\begin{aligned} \mathcal{P}(\Omega_k) &\leq \mathcal{P}(\Omega_k^{(1)}) + \mathcal{P}(\Omega_k^{(2)}) \leq \mathcal{P}(\Omega_k) + 2\mathcal{H}^{n-1}(\Omega_k \cap \partial B_{r_k}) \\ &\stackrel{(3.39)}{\leq} \mathcal{P}(\Omega_k) + \frac{8V}{k}. \end{aligned} \quad (3.40)$$

Moreover, since $|\Omega_k| = V$ for all k , (3.37) translates into

$$\chi_{\Omega_k^{(1)}} \rightarrow \chi_{\Omega} \quad \text{and} \quad \chi_{\Omega_k^{(2)}} \rightarrow 0 \quad \text{in } L^2(\mathbb{R}^n) \text{ for } k \rightarrow \infty.$$

Hence $\nabla \Phi_{\Omega_k^{(1)}} \rightarrow \nabla \Phi_{\Omega}$ and $\nabla \Phi_{\Omega_k^{(2)}} \rightarrow 0$ in $L^2(\mathbb{R}^n; \mathbb{R}^n)$ and

$$\mathcal{N}(\Omega_k) - \mathcal{N}(\Omega_k^{(1)}) - \mathcal{N}(\Omega_k^{(2)}) \stackrel{(3.4)}{=} I(\Omega_k^{(1)}, \Omega_k^{(2)}) \rightarrow 0 \text{ for } k \rightarrow \infty. \quad (3.41)$$

Combining (3.40) and (3.41) yields (3.36) and completes the proof. \square

3.5 $C^{1,\alpha}$ -regularity

In this section we use the regularity theory for quasi-minimizers of the perimeter functional to prove $C^{1,\alpha}$ -regularity of regular local minimizers of \mathcal{E} . Such results are by now classical and our proof is essentially a combination of standard arguments. From the numerous results of this type in the literature,

the following two seem to be closest to our setting. In [5], Anzellotti, Baldo and Visintin study the energy functional \mathcal{E} restricted to a bounded domain. They establish $C^{1,\alpha}$ regularity for suitable representatives of global minimizers without volume constraint. In [82], Rigot proves $C^{1,\alpha}$ regularity for certain representatives of global minimizers of a related (but more regular) energy functional subject to a volume constraint.

First, we recall the definition of a quasi-minimizer of perimeter.

Definition 3.5.1 (Quasi-minimizer of perimeter). Let $\omega : (0, R) \rightarrow (0, \infty)$ be an increasing function with $\lim_{r \rightarrow 0^+} \omega(r) = 0$. A measurable set $F \subset \mathbb{R}^n$ is called a quasi-minimizer of perimeter (with respect to ω) if

$$\mathcal{P}(F; B_r(x)) \leq \mathcal{P}(G; B_r(x)) + \omega(r)r^{n-1} \quad (3.42)$$

for all $x \in \text{supp } D\chi_F$, all $r \in (0, R)$ and all measurable $G \subset \mathbb{R}^n$ with $F \Delta G \subset \subset B_r(x)$.

Our main tool (in this section) is the regularity result for quasi-minimizers of perimeter due to Tamanini [91], extending earlier results due to De Giorgi [25]. It states that quasi-minimizers of perimeter enjoy the following regularity properties

Theorem 3.5.2 (Tamanini, [91, Theorem 1, Lemma 4]). *Let $2 \leq n \leq 7$ and let $\omega : (0, R) \rightarrow (0, \infty)$ be given by $\omega(r) = Cr^{2\alpha}$ for some $\alpha \in (0, \frac{1}{2})$ and $C, R > 0$. If $F \subset \mathbb{R}^n$ is a quasi-minimizer for perimeter, then $\text{supp } D\chi_F = \partial^* F$ is a $C^{1,\alpha}$ -hypersurface and the $C^{1,\alpha}$ -constants only depend on C, R and α . Moreover, there is $\bar{R} > 0$ such that the following density bounds*

$$\omega_{n-1}r^n \left(1 - \frac{1}{2n}\right) \geq |F \cap B_r(x)| \geq \omega_{n-1} \frac{r^n}{2n} \quad (3.43)$$

hold for all $x \in \text{supp } D\chi_F$ and all $r \in (0, \bar{R})$, where $\bar{R} > 0$ depends on ω and n and ω_{n-1} denotes the volume of the $(n-1)$ -ball.

The main result of this section is the following proposition. It constitutes the first step towards the proof of Theorem 3.2.3.

Proposition 3.5.3 ($C^{1,\alpha}$ -Regularity for local minimizers). *Let $2 \leq n \leq 7$ and let $\Omega \in \mathcal{C}_V$ be a regular local minimizer of \mathcal{E} . Then Ω is an open bounded set with $C^{1,\alpha}$ boundary for every $\alpha \in (0, \frac{1}{2})$.*

Our strategy to prove Proposition 3.5.3 is to show that every local minimizer $\Omega \in \mathcal{C}_V$ of \mathcal{E} satisfies (3.42) with $\omega(r) = Cr^{2\alpha}$ and then to apply Theorem 3.5.2. Adapting a standard technique (see, e.g., [67, p.279]), we remove the volume constraint and show that Ω is an unconstrained minimizer of a suitable penalized energy functional. If we knew that $\partial_1 \Phi_\Omega \in L^\infty(\mathbb{R}^n)$ quasi-minimality of Ω would easily follow from the estimate

$$|\mathcal{N}(\Omega) - \mathcal{N}(F)| \leq (1 + 2\|\partial_1 \Phi_\Omega\|_{L^\infty})|F\Delta\Omega|.$$

But we have not yet shown that $\partial_1 \Phi_\Omega \in L^\infty(\mathbb{R}^n)$ for minimizers Ω (which we will obtain as a consequence of Proposition 3.5.3). To avoid circular reasoning, we work with estimate (3.23) instead.

Proof of Proposition 3.5.3. Let $\Omega \in \mathcal{C}_V$ be a regular local minimizer of \mathcal{E} .

Step 1: We claim that there is some $\Lambda_\Omega > 0$ such that Ω minimizes the unconstrained problem

$$J_\Omega(F) := \mathcal{E}(F) + \Lambda_\Omega |F\Delta\Omega|$$

among all $F \subset \mathbb{R}^n$ with finite perimeter. We argue by contradiction and assume that there is a sequence $\Lambda_k \rightarrow \infty$ and sets $F_k \subset \mathbb{R}^n$ such that

$$\mathcal{E}(F_k) + \Lambda_k |F_k\Delta\Omega| < \mathcal{E}(\Omega). \quad (3.44)$$

Setting $\widehat{F}_k := \mu_k F_k$ where $\mu_k := \left(\frac{|\Omega|}{|F_k|}\right)^{\frac{1}{n}}$, we have $|\widehat{F}_k| = |\Omega|$ for all k . Moreover, $\Lambda_k \rightarrow \infty$ and (3.44) imply $\mu_k \rightarrow 1$ and

$$|\Omega\Delta\widehat{F}_k| \leq |\Omega\Delta\mu_k\Omega| + \mu_k^n |\Omega\Delta F_k| \rightarrow 0.$$

Hence, for sufficiently large k , \widehat{F}_k is an admissible competitor to Ω and local minimality of Ω implies

$$\mathcal{E}(\Omega) \leq \mathcal{E}(\widehat{F}_k) \quad \text{for all sufficiently large } k. \quad (3.45)$$

On the other hand, we deduce from the scaling properties of \mathcal{E} and (3.44) that

$$\begin{aligned} \mathcal{E}(\widehat{F}_k) &= \mathcal{E}(F_k) + (\mu_k^{n-1} - 1)\mathcal{P}(F_k) + (\mu_k^n - 1)\mathcal{N}(F_k) \\ &\stackrel{(3.44)}{\leq} \mathcal{E}(\Omega) - \Lambda_k |F_k \Delta \Omega| + \max\{|\mu_k^{n-1} - 1|, |\mu_k^n - 1|\} \mathcal{E}(\Omega). \end{aligned} \quad (3.46)$$

Note that (3.45) and (3.46) imply $\mu_k \neq 1$ for sufficiently large k . Inserting $|F_k \Delta \Omega| \geq ||F_k| - |\Omega|| = |\mu_k^{-n} - 1||\Omega|$ into (3.46), we obtain

$$\mathcal{E}(\widehat{F}_k) \leq \mathcal{E}(\Omega) + |\mu_k^{-n} - 1| \left(-\Lambda_k |\Omega| + \frac{\max\{|\mu_k^{n-1} - 1|, |\mu_k^n - 1|\}}{|\mu_k^{-n} - 1|} \mathcal{E}(\Omega) \right).$$

Since $\mu_k \rightarrow 1$ and $\Lambda_k \rightarrow \infty$, the expression in parentheses becomes negative for sufficiently large k which contradicts (3.45) and proves the claim of Step 1.

Step 2: We show that Ω is a quasi-minimizer of perimeter with ω of the form $\omega(r) = Cr^{2\alpha}$. Let $R > 0$ and let $F \Delta \Omega \subset\subset B_r(x)$ for some $r \leq R$. Then minimality of Ω for J_Ω yields

$$\mathcal{P}(\Omega) \leq \mathcal{P}(F) + \Lambda_\Omega |F \Delta \Omega| + \mathcal{N}(F) - \mathcal{N}(\Omega).$$

Exploiting continuity of \mathcal{N} by applying (3.23) with $\frac{1}{q} := \frac{n-1+2\alpha}{n}$, we get

$$\begin{aligned} \mathcal{P}(\Omega) &\leq \mathcal{P}(F) + \Lambda_\Omega |F \Delta \Omega| + C(\alpha, n) |\Omega \cup F|^{\frac{1-2\alpha}{n}} |\Omega \Delta F|^{\frac{n-1+2\alpha}{n}} \\ &\leq \mathcal{P}(F) + C(\alpha, R, \Omega) r^{n-1+2\alpha} \quad \text{for } r \leq R \end{aligned}$$

where $C(\alpha, R, \Omega) := \Lambda_\Omega \omega_n^2 R^{1-2\alpha} + C(\alpha, n) \omega_n (2|\Omega| + \omega_n R^n)^{\frac{1-2\alpha}{n}}$. Hence, Ω is a quasi-minimizer of perimeter and Theorem 3.5.2 yields the $C^{1,\alpha}$ -regularity of $\text{supp } D\chi_\Omega = \partial^* \Omega$ and the density bounds.

Step 3: We use the density estimates (3.43) to show that Ω is bounded. Assume for contradiction that Ω is not bounded. Then there is a sequence $(x_k)_{k \in \mathbb{N}}$ of points in Ω with $|x_k - x_j| \geq \bar{R}$ for all $j \neq k$. Then the lower bound on the density implies

$$|\Omega \cap B_{\bar{R}}(x_k)| \geq \frac{\omega_{n-1}}{2n} \left(\frac{\bar{R}}{2} \right)^n \quad \text{for all } k$$

and hence

$$|\Omega| \geq \sum_k |\Omega \cap B_{\bar{R}/2}(x_k)| = +\infty$$

which contradicts $|\Omega| < \infty$. Thus, such a sequence $(x_k)_{k \in \mathbb{N}}$ cannot exist and Ω must be bounded.

Step 4: We show that Ω is open with $C^{1,\alpha}$ boundary. To this end, we introduce the open sets

$$\begin{aligned}\Omega_{(1)} &:= \{x \in \mathbb{R}^n : \exists r > 0 \text{ s.t. } |\Omega \cap B_r(x)| = |B_r|\}, \\ \Omega_{(0)} &:= \{x \in \mathbb{R}^n : \exists r > 0 \text{ s.t. } |\Omega \cap B_r(x)| = 0\}.\end{aligned}$$

We first show that $\Omega = \Omega_{(1)}$. Indeed, it is straightforward to check (using the definitions) that

$$\Omega_{(1)} \subset \mathring{\Omega}^M \subset \Omega_{(1)} \cup (\text{supp } D\chi_\Omega \setminus \partial^*\Omega). \quad (3.47)$$

Since $\text{supp } D\chi_\Omega = \partial^*\Omega$ by Step 2 and $\Omega = \mathring{\Omega}^M$, (3.47) implies $\Omega = \Omega_{(1)}$. Likewise, it is straightforward to check that $\partial(\Omega_{(1)}) \subset \mathbb{R}^n \setminus (\Omega_{(1)} \cup \Omega_{(0)})$. Since $\text{supp } D\chi_\Omega \subset \partial\Omega$ we get

$$\partial\Omega = \partial(\Omega_{(1)}) = \mathbb{R}^n \setminus (\Omega_{(1)} \cup \Omega_{(0)}) = \text{supp } D\chi_\Omega.$$

Since $\text{supp } D\chi_\Omega$ is a $C^{1,\alpha}$ -hypersurface (see Step 2), we conclude that Ω is a bounded open set with $C^{1,\alpha}$ boundary. \square

3.6 Higher regularity

In this section, we prove Theorems 3.2.3 and 3.2.5 and Corollary 3.2.4. We proceed in two steps: In Proposition 3.6.2 (below), we exploit stationarity of \mathcal{E} with respect to inner variations to show that the weak Noether equation (3.16) holds for minimizers. In Proposition 3.6.3 (below), we use a bootstrap argument for (3.16) to deduce smoothness of the boundary and $\langle \partial_1 \Phi_\Omega \rangle$. Theorem 3.2.3 is then a direct consequence of Propositions 3.5.3, 3.6.2 and 3.6.3. In turn, Corollary 3.2.4 follows by an application of Theorem 3.3.3.

3.6.1 Computation of the inner variation of \mathcal{E}

We present the derivation of the weak Noether equation for regular local minimizers of \mathcal{E} . First, we introduce the necessary notation. Let $T \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$

be a compactly supported smooth vector field and let $\{F_t\}_{|t|<t_0}$ be a family of diffeomorphisms $F_t : \mathbb{R}^n \rightarrow \mathbb{R}^n$ for all $t \in (-t_0, t_0)$. We call $\{F_t\}_{|t|<t_0}$ a local variation with initial velocity T , if

- (i) $(x, t) \mapsto F_t(x)$ is smooth on $\mathbb{R}^n \times (-t_0, t_0)$,
- (ii) $F_0 = \text{Id}_{\mathbb{R}^n}$ and $\partial_t F_t \Big|_{t=0} = T$,
- (iii) there is $R > 0$ with $\{F_t(x) \neq x\} \subset B_R$ for all $|t| < t_0$.

To simplify the notation, we write $\Omega_t := F_t(\Omega)$.

We compute the first variation of \mathcal{E} at sufficiently regular sets. Our approach is based on a diffuse interface approximation in order to regularize the nonlocal term in the energy. Before we present our argument (see Lemma 3.6.1 below), let us explain why such a regularization procedure seems necessary. It might look promising to define

$$\Psi(s, t) = \int_{\mathbb{R}^n} \nabla \Phi_{\Omega_s} \cdot \nabla \Phi_{\Omega_t} \, d\mathcal{H}^{n-1} = \int_{\Omega_t} \partial_1 \Phi_{\Omega_s} \, d\mathcal{H}^{n-1}$$

and to compute $\frac{d}{dt} \Big|_{t=0} \mathcal{N}(\Omega_t) = \frac{d}{dt} \Big|_{t=0} \Psi(t, t)$. If Ψ was differentiable, we would have $\frac{d}{dt} \Big|_{t=0} \Psi(t, t) = 2\partial_1 \Psi(0, 0)$ by symmetry of Ψ . Hence, the (usually simpler) computation of the partial derivative $\partial_1 \Psi(0, 0)$ would be sufficient. However, a computation shows that, in general, Ψ is merely Lipschitz continuous in each component. The one-sided derivatives $\partial_1^+ \Psi$ and $\partial_1^- \Psi$ differ on the diagonal $\{(t, t) : t \in \mathbb{R}\}$ for suitable initial velocities. This is of course related to the fact that $\partial_1 \Phi_{\Omega_t}$ jumps on $\partial\Omega_t$.

Lemma 3.6.1. *Let Ω be open and bounded with $C^{1,\alpha}$ boundary for some $\alpha > 0$ and $\{F_t\}_{|t|<t_0}$ a local variation with initial velocity $T \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$. Then the first inner variation $\partial\mathcal{E}(\Omega, T)$ of \mathcal{E} at Ω in direction T is given by*

$$\begin{aligned} \partial\mathcal{E}(\Omega, T) &:= \frac{d}{dt} \Big|_{t=0} \mathcal{E}(F_t(\Omega)) \\ &= \int_{\partial\Omega} \text{div}_\Omega T \, d\mathcal{H}^{n-1} + \int_{\partial\Omega} 2\langle \partial_1 \Phi_\Omega \rangle (T \cdot \nu) \, d\mathcal{H}^{n-1}. \end{aligned} \tag{3.48}$$

Proof. It is well-known that the first variation of the perimeter is given by (see, e.g., [67, Theorem 17.5])

$$\frac{d}{dt} \Big|_{t=0} \mathcal{P}(\Omega_t) = \int_{\partial\Omega} \text{div}_\Omega T \, d\mathcal{H}^{n-1}.$$

Hence, it remains to show that

$$\left. \frac{d}{dt} \right|_{t=0} \mathcal{N}(\Omega_t) = \int_{\partial\Omega} 2\langle \partial_1 \Phi_\Omega \rangle (T \cdot \nu) \, d\mathcal{H}^{n-1}. \quad (3.49)$$

Let $\rho \in C_c^\infty(\mathbb{R}^n)$ denote a rotational symmetric mollifier and set $\rho_\varepsilon := \frac{1}{\varepsilon^n} \rho(\frac{\cdot}{\varepsilon})$ for all $\varepsilon > 0$. We introduce

$$u_{\varepsilon,t} := \rho_\varepsilon * \chi_{\Omega_t}$$

and the corresponding field $\nabla \Phi_{\varepsilon,t}$ as the unique weak solution $\Phi_{\varepsilon,t} \in \dot{H}^1(\mathbb{R}^n)$ of

$$\Delta \Phi_{\varepsilon,t} = \partial_1 u_{\varepsilon,t}. \quad (3.50)$$

To simplify the notation, we set $X_t := \partial_t F_t \circ F_t^{-1}$ and introduce the functions $f_\varepsilon, g : (-t_0, t_0) \rightarrow \mathbb{R}$, given by

$$f_\varepsilon(t) := \int_{\mathbb{R}^n} |\nabla \Phi_{\varepsilon,t}|^2 \, dx \quad \text{and} \quad (3.51)$$

$$g(t) := \int_{\partial\Omega_t} 2\langle \partial_1 \Phi_{\Omega_t} \rangle (X_t \cdot \nu) \, dx. \quad (3.52)$$

We claim that there is $t_* \in (0, t_0)$ such that $f_\varepsilon \in C^1([-t_*, t_*])$ and

$$\sup_{t \in [-t_*, t_*]} |f'_\varepsilon(t) - g(t)| \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \quad (3.53)$$

Assuming for a moment that (3.53) holds, the proof closes as follows. Using L^2 -estimates for the elliptic equation (3.50) we see that $f_\varepsilon(t) \rightarrow \mathcal{N}(\Omega_t)$ as $\varepsilon \rightarrow 0$ pointwise for all $t \in (-t_0, t_0)$. Hence, (3.53) implies

$$\frac{d}{dt} \mathcal{N}(\Omega_t) = g \quad \text{on } (-t_*, t_*).$$

Evaluating $g(0)$ using $X_0 = T$ then yields the claim (3.49).

We turn to the computation of f'_ε . Note that $\tilde{F}_s := F_{t+s} \circ F_t^{-1}$ is a local variation with initial velocity $X_t := \partial_t F_t \circ F_t^{-1}$. A standard computation (see, e.g., [67, Prop. 17.8]) yields

$$\partial_t u_{\varepsilon,t}(x) = \int_{\partial\Omega_t} \rho_\varepsilon(x-y) (X_t(y) \cdot \nu(y)) \, d\mathcal{H}^{n-1}(y). \quad (3.54)$$

Since $u_{\varepsilon,t}$ is continuously differentiable in t and $\text{supp } u_{\varepsilon,t} \subset\subset B_R$ for all $t \in (-t_0, t_0)$ and some $R > 0$, we have for any $\varepsilon > 0$ fixed

$$\lim_{h \rightarrow 0} \frac{u_{\varepsilon,t+h} - u_{\varepsilon,t}}{h} = \partial_t u_{\varepsilon,t} \quad \text{in } L^2(\mathbb{R}^n). \quad (3.55)$$

Moreover, $\partial_t u_{\varepsilon,t}$ is bounded in $L^2(\mathbb{R}^n)$ uniformly in $t \in (-t_0, t_0)$. Let $S : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n; \mathbb{R}^n)$ be given by $S(f) := \nabla(\partial_1 \Gamma * f)$ (cf. (3.17)), so that in particular $S(u_{\varepsilon,t}) = \nabla \Phi_{\varepsilon,t}$. Since S is linear and L^2 -continuous, a short computation using (3.55) yields

$$f'_\varepsilon(t) = 2 \int_{\mathbb{R}^n} S(u_{\varepsilon,t}) \cdot S(\partial_t u_{\varepsilon,t}) \, dx.$$

We remove S again by inserting the equations $S(u_{\varepsilon,t}) = \nabla \Phi_{\varepsilon,t}$ and $\nabla \cdot S(\partial_t u_{\varepsilon,t}) = \partial_1 \partial_t u_{\varepsilon,t}$ and integrating by parts twice. We obtain

$$f'_\varepsilon(t) = 2 \int_{\mathbb{R}^n} \partial_1 \Phi_{\varepsilon,t} \partial_t u_{\varepsilon,t} \, dx. \quad (3.56)$$

Upon inserting the representation $\partial_1 \Phi_{\varepsilon,t} = (\partial_1 \Gamma * \partial_1(\rho_\varepsilon * \chi_{\Omega_t}))$ and (3.54) into (3.56), we conclude that

$$\begin{aligned} f'_\varepsilon(t) &= 2 \int_{\mathbb{R}^n} (\partial_1 \Gamma * \partial_1(\rho_\varepsilon * \chi_{\Omega_t})) \partial_t u_{\varepsilon,t} \, dx \\ &= 2 \int_{\partial \Omega_t} (\partial_1 \Gamma * \partial_1(\tilde{\rho}_\varepsilon * \chi_{\Omega_t}))(x) (X_t(x) \cdot \nu(x)) \, d\mathcal{H}^{n-1}(x), \end{aligned} \quad (3.57)$$

where we have abbreviated $\tilde{\rho}_\varepsilon = \rho_\varepsilon * \rho_\varepsilon$. It remains to investigate the limit

$$\lim_{\varepsilon \rightarrow 0} (\partial_1 \Gamma * \partial_1(\tilde{\rho}_\varepsilon * \chi_{\Omega_t}))(x) = - \lim_{\varepsilon \rightarrow 0} \int_{\partial \Omega_t} (\tilde{\rho}_\varepsilon * \partial_1 \Gamma)(x - y) \nu_1(y) \, d\mathcal{H}^{n-1}(y).$$

Note that when the family of regularized kernels $\tilde{\rho}_\varepsilon * \partial_1 \Gamma$ in the last expression is replaced by the truncated kernels $\chi_{\mathbb{R}^n \setminus B_\varepsilon} \partial_1 \Gamma$, we have

$$- \lim_{\varepsilon \rightarrow 0} \int_{\partial \Omega_t \setminus B_\varepsilon(x)} \partial_1 \Gamma(x - y) \nu_1(y) \, d\mathcal{H}^{n-1}(y) = \langle \partial_1 \Phi_{\Omega_t} \rangle(x).$$

By means of a lengthy but conceptually simple estimate (which we defer to Lemma B.3 in the appendix) we obtain the uniform convergence

$$(\partial_1 \Gamma * \partial_1(\tilde{\rho}_\varepsilon * \chi_{\Omega_t}))(x) \xrightarrow{\varepsilon \rightarrow 0} \langle \partial_1 \Phi_{\Omega_t} \rangle(x) \quad \text{for all } x \in \Omega_t \quad (3.58)$$

uniformly in $t \in [-t_*, t_*]$ and x . Since $\partial \Omega_t$ is compact and uniformly bounded in t , applying (3.58) to (3.57) proves (3.53). This completes the proof. \square

We now show that regular local minimizers of \mathcal{E} satisfy the weak Noether equation.

Proposition 3.6.2. *Let $2 \leq n \leq 7$ and let Ω be a regular local minimizer of \mathcal{E} . Then the weak Noether equation (3.16) holds on $\partial\Omega$.*

Our proof is based on Lemma 3.6.1 and a combination of well-known arguments regarding the existence of a Lagrange multiplier (see, e.g., [36, p.90] and [67, p.208]). However, we have not been able to find a reference that applies precisely to our setting. For the convenience of the reader, we provide the details below.

Proof of Proposition 3.6.2. Let Ω be a regular local minimizer of \mathcal{E} . Then Ω is an open, bounded set with $C^{1,\alpha}$ boundary by Prop. 3.5.3. Our goal is to construct a local variation $\{G_t\}_{|t| \leq t_0}$ with $|G_t(\Omega)| = |\Omega|$ for $|t| < t_0$.

Let $\{F_t\}_{|t| \leq t_0}$ be a local variation with initial velocity $T \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$. It is well-known that the first variation of Lebesgue measure is given by (see, e.g., [67, p.202])

$$\left. \frac{d}{dt} \right|_{t=0} |F_t(\Omega)| = \int_{\partial\Omega} T \cdot \nu \, d\mathcal{H}^{n-1} = \int_{\Omega} \operatorname{div} T \, dx. \quad (3.59)$$

We consider vector fields $S, T \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$ where T is arbitrary and S satisfies

$$\int_{\partial\Omega} S \cdot \nu \, d\mathcal{H}^{n-1} \neq 0. \quad (3.60)$$

Define the smooth family of diffeomorphisms

$$F_{s,t} \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n) \quad F_{s,t}(x) = x + sS(x) + tT(x)$$

for $s, t \in (-r, r)$ with $r > 0$ sufficiently small and consider the smooth function $f(s, t) = |F_{s,t}(\Omega)|$. Since $\{\tilde{F}_h\}_{|h| < r}$ with $\tilde{F}_h := F_{h+s,t} \circ F_{s,t}^{-1}$ is a local variation with initial velocity $\partial_h \tilde{F}_h \Big|_{h=0} = \partial_s F_{s,t} \circ F_{s,t}^{-1}$, (3.59) implies

$$\nabla f(s, t) = \begin{pmatrix} \int_{F_{s,t}(\Omega)} \operatorname{div}(\partial_s F_{s,t} \circ F_{s,t}^{-1}) \, dx \\ \int_{F_{s,t}(\Omega)} \operatorname{div}(\partial_t F_{s,t} \circ F_{s,t}^{-1}) \, dx \end{pmatrix}.$$

Moreover, (3.60) shows that $\partial_s f(0,0) \neq 0$. Thus, upon possibly reducing $r > 0$, the implicit function theorem yields a function $s : (-r, r) \rightarrow \mathbb{R}$ with

$$f(s(t), t) = f(0, 0) \quad \text{for } t \in (-r, r) \quad \text{and} \quad s'(0) = -\frac{\int_{\partial\Omega} T \cdot \nu \, d\mathcal{H}^{n-1}}{\int_{\partial\Omega} S \cdot \nu \, d\mathcal{H}^{n-1}}. \quad (3.61)$$

In particular, $G_t := F_{s(t), t}$ is a local variation with initial velocity $s'(0)S + T$ and it satisfies $|G_t(\Omega)| = f(0, 0) = |\Omega|$ for all $t \in (-r, r)$. Since Ω is of finite perimeter we also have (see, e.g., [67, Lemma 17.9])

$$|G_t(\Omega) \Delta \Omega| \rightarrow 0 \quad \text{for } t \rightarrow 0.$$

Hence $G_t(\Omega)$ is an admissible comparison set to Ω for sufficiently small $|t|$. Local minimality of Ω hence requires

$$\partial\mathcal{E}(\Omega, s'(0)S + T) \stackrel{(3.48)}{=} \left. \frac{d}{dt} \right|_{t=0} \mathcal{E}(G_t(\Omega)) = 0. \quad (3.62)$$

By linearity of the first variation in the initial velocity (see (3.48)) and upon inserting (3.61) into (3.62), we conclude

$$\partial\mathcal{E}(\Omega, T) = \Lambda \int_{\partial\Omega} T \cdot \nu \, d\mathcal{H}^{n-1} \quad \text{where} \quad \Lambda = \frac{\partial\mathcal{E}(\Omega, S)}{\int_{\partial\Omega} S \cdot \nu \, d\mathcal{H}^{n-1}}.$$

Hence, the weak Noether equation (3.16) holds and the proof of Proposition 3.6.2 is complete. \square

3.6.2 Higher regularity for regular critical points

In this section, we use a bootstrap argument for (3.16) to deduce higher regularity of $\partial\Omega$ and $\langle \nabla \Phi_\Omega \rangle$.

Proposition 3.6.3 (Regular critical points are smooth). *Let Ω be a regular critical point. Then the following holds*

(i) *The boundary $\partial\Omega$ is a smooth hypersurface.*

(ii) *The Noether equation holds in the strong form*

$$H_\Omega + 2\langle \partial_1 \Phi_\Omega \rangle = \Lambda \quad \text{on } \partial\Omega.$$

Proof of Proposition 3.6.3. Fix $x_0 \in \partial\Omega$. By Definition 3.2.6 (i) and upon rotation and translation we may assume $x_0 = 0$ and

$$\partial\Omega \cap (D \times (\varepsilon, \varepsilon)) = \text{graph}(u),$$

where D denotes a ball in \mathbb{R}^{n-1} with radius $r > 0$, $u \in C^{1,\alpha}(D)$ and $\varepsilon > 0$. Let $\varphi \in C_c^\infty(D)$ and $\eta \in C_c^\infty(\mathbb{R})$ with $\eta \equiv 1$ on $(-\varepsilon, \varepsilon)$. Testing (3.16) with $T(x) = \eta(x_n)\varphi(x')e_n$ yields

$$-\int_D \frac{\nabla' u \cdot \nabla' \varphi}{\sqrt{1 + |\nabla' u|^2}} dx' + 2 \int_D \langle \partial_1 \Phi_\Omega \rangle (x' + u(x')e_n) \varphi dx' = \Lambda \int_D \varphi dx'$$

for all $\varphi \in C_c^\infty(D)$. This is the weak formulation of the following elliptic equation for u

$$\text{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = -2 \langle \partial_1 \Phi_\Omega \rangle (\text{Id}, u) + \Lambda. \quad (3.63)$$

We use a bootstrap argument to show that the solution of (3.63) is smooth. It is based on the following two implications, which hold for every integer $k \geq 0$ and $\alpha \in (0, 1)$.

- (i) When $u \in C^{k+1,\alpha}(D)$, then Theorem 3.3.3 (iii) implies $\langle \nabla \Phi_\Omega \rangle \in C^{k,\alpha}(\partial\Omega)$ and the right hand side of (3.63) is in $C^{k,\alpha}(D)$.
- (ii) When the right hand side of (3.63) is in $C^{k,\alpha}(D)$ then Schauder Theory yields $u \in C^{k+2,\alpha}(D)$ (see, e.g., [37, Theorem 9.19]).

Hence, $\partial\Omega$ is a smooth hypersurface and $\langle \nabla \Phi_\Omega \rangle \in C^\infty(\partial\Omega; \mathbb{R}^n)$. In particular $\partial\Omega \in C^2$ which allows rewrite the first variation of the perimeter as (see, e.g., [67, Remark 17.7])

$$\int_{\partial\Omega} \text{div}_\Omega T \, dx = \int_{\partial\Omega} H_\Omega (T \cdot \nu) \, d\mathcal{H}^{n-1}, \quad (3.64)$$

where H_Ω denotes the sum of the principal curvatures. Hence (3.64) and (3.16) imply (3.15). \square

3.7 Topological properties of minimizers

In this section, we give the proof of Theorem 3.2.7, which is organized as follows. Connectedness of a regular local minimizer Ω is proved in Proposition

3.7.1 below. Since \mathcal{E} is invariant with respect to taking complements, connectedness of $\mathbb{R}^n \setminus \overline{\Omega}$ is obtained by essentially the same argument in Lemma 3.7.2. In turn, we invoke a topological Lemma to deduce connectedness of $\partial\Omega$ from connectedness of Ω and $\mathbb{R}^n \setminus \overline{\Omega}$. Finally, for $n = 2$ we use the Jordan-Schoenflies Theorem to conclude that regular local minimizers are topologically equivalent to a ball. Theorem 3.2.7 is then an immediate consequence of Proposition 3.7.1 and Lemmas 3.7.2–3.7.4.

Proposition 3.7.1. *Let $2 \leq n \leq 7$ and let $\Omega \in \mathcal{C}_V$ be a regular local minimizer of \mathcal{E} . Then Ω is connected.*

Proof of Proposition 3.7.1. Assume for contradiction that Ω is not connected and thus can be written as the union $\Omega = A \dot{\cup} B$ of two nonempty disjoint open sets A, B . We will show

Step 1: The regularity of $\partial\Omega$ implies

$$\text{dist}(A, B) > 0. \quad (3.65)$$

Step 2: Let $Z_Y(0)$ denote the connected component of 0 in the nonempty open set $Y := \{y \in \mathbb{R}^n \mid \text{dist}(A, y + B) > 0\}$. Then $y \mapsto \mathcal{E}(A \cup (y + B))$ is constant on $Z_Y(0)$.

In case Ω is a global minimizer, we are almost done: We can find $\bar{y} \in \partial Y \cap Z_Y(0)$ such that $A \cup (\bar{y} + B)$ is another global minimizer. However, $A \cup (\bar{y} + B)$ violates (3.65) and thus yields the desired contradiction. The general case of a local minimizer requires a bit more work.

Step 3: Evaluating the Noether equation for a family of local minimizers constructed with the help of Step 2, we obtain $\partial_1 \Phi_B \equiv 0$ and hence $|B| = 0$.

Obviously, $|B| = 0$ contradicts our assumption that B is open and nonempty. Hence Ω must be connected and the proof is complete. We turn to the proof of Steps 1–3.

Proof of Step 1: If (3.65) was false then boundedness of Ω implies that there is a point $x \in \overline{A} \cap \overline{B} \subset \overline{\Omega}$. Since Ω is open with smooth boundary, the open set $B_\varepsilon(x) \cap \Omega$ is connected for sufficiently small $\varepsilon > 0$. This contradicts the fact that $B_\varepsilon(x) \cap \Omega$ contains points from A and B . Hence (3.65) holds.

Proof of Step 2: We set $\Omega_y := A \cup (y + B)$ and investigate how the energy

$\mathcal{E}(\Omega_y)$ changes for y in the open set

$$Y = \{y \in \mathbb{R}^n \mid \text{dist}(A, y + B) > 0\}.$$

As \mathcal{E} is invariant under translation, only the interaction energy changes. Also using linearity of (3.2), we get

$$\mathcal{E}(\Omega_y) - \mathcal{E}(\Omega) = 2 \int_{\mathbb{R}^n} \nabla \Phi_A \cdot (\nabla \Phi_{y+B} - \nabla \Phi_B) \, dx. \quad (3.66)$$

To simplify the notation, we neglect the constant terms and the factor 2 in (3.66) and introduce

$$f : Y \rightarrow \mathbb{R}, \quad f(y) = \int_{\mathbb{R}^n} \nabla \Phi_A \cdot \nabla \Phi_{y+B} \, dx.$$

Inserting the identity $\Phi_{y+B}(x) = \Phi_B(x - y)$ and using the weak formulation of (3.2) and a translation by y allows to rewrite f as

$$f(y) = \int_B \partial_1 \Phi_A(x + y) \, dx. \quad (3.67)$$

We claim that f is harmonic. Indeed for any $y_0 \in Y$ we can find $\overline{B}_\varepsilon(y_0) \subset Y$ and some constant $c_\varepsilon > 0$ such that $\text{dist}(\overline{A}, y + B) \geq c_\varepsilon$ for all $y \in \overline{B}_\varepsilon(y_0)$. Since $\partial_1 \Phi_A$ is harmonic on $\mathbb{R}^n \setminus \partial A$ we may deduce that f is harmonic on $\overline{B}_\varepsilon(y_0)$ by differentiating under the integral sign in (3.67). Since Ω is a local minimizer of \mathcal{E} , we have $f(y) \geq f(0)$ for all y in a suitable neighborhood of 0. The strong maximum principle asserts that f is constant on $Z_Y(0)$, the connected component of Y which contains 0.

Proof of Step 3: Since Ω is a local minimizer, there is $\delta > 0$ such that

$$\mathcal{E}(F) \geq \mathcal{E}(\Omega) \quad \text{for all } F \in \mathcal{C}_V \text{ with } |F \Delta \Omega| < \delta.$$

We claim that there is $\varepsilon > 0$ such that for all $y \in B_\varepsilon$ the set $\Omega_y = A \cup (y + B)$ is a local minimizer. Indeed, since B is of finite measure, translating B is a continuous operation and hence there is $\varepsilon > 0$ such that $|B \cup (y + B)| < \delta/2$ for all $y \in B_\varepsilon$. By possibly reducing ε we may assume $B_\varepsilon \subset Z_Y(0)$ from Step 2. Moreover, for every $y \in B_\varepsilon$ and every $F \in \mathcal{C}_V$ with $|F \Delta \Omega_y| < \frac{\delta}{2}$, the triangle inequality implies

$$|F \Delta \Omega| \leq |F \Delta \Omega_y| + |B \Delta (y + B)| < \delta.$$

and hence

$$\mathcal{E}(F) \geq \mathcal{E}(\Omega) \stackrel{\text{Step 2}}{=} \mathcal{E}(\Omega_y).$$

Theorem 3.2.3 implies that the Noether equation

$$H_{\Omega_y} + 2\langle \partial_1 \Phi_{\Omega_y} \rangle = \Lambda_y \quad \text{on } \partial\Omega_y \quad (3.68)$$

holds for some $\Lambda_y \in \mathbb{R}$. We will evaluate the Noether equation at two suitable points $p, q \in \partial\Omega_y$ to show that $|B| = 0$. The choice of the p, q will become clear later. Let $p \in \partial\Omega$ be such that

$$\sup_{x \in \overline{\Omega}} x_1 = p_1$$

and assume w.l.o.g that $p \in \partial A$. Furthermore, let $q \in \partial A$ satisfy $q = p - te_1$ with minimal t . Evaluating (3.68) at p and q and using linearity of (3.2), we obtain

$$\begin{aligned} & H_A(p) + 2\langle \partial_1 \Phi_A \rangle(p) + 2\partial_1 \Phi_B(p - y) \\ &= H_A(q) + 2\langle \partial_1 \Phi_A \rangle(q) + 2\partial_1 \Phi_B(q - y). \end{aligned}$$

This shows that $y \mapsto \partial_1 \Phi_B(q - y) - \partial_1 \Phi_B(p - y)$ is constant on B_ε . Since it is harmonic, it is even constant on the connected component of 0 in $\mathbb{R}^n \setminus ((p - B) \cup (q - B))$ and, in particular, on $L := \{te_1 : t \geq 0\}$. Since $\partial_1 \Phi_B$ decays at infinity, we conclude

$$\partial_1 \Phi_B(q - y) = \partial_1 \Phi_B(p - y) \quad \text{for all } y \in L.$$

This means that on $q + L$, $\partial_1 \Phi_B$ equals a periodic function. Using the decay at infinity again, we get $\partial_1 \Phi_B \equiv 0$ on $q + L$. However, the asymptotic behavior of $\partial_1 \Phi_B$ is given by

$$\begin{aligned} \partial_1 \Phi_B(q + te_1) &= \int_{\mathbb{R}^n} \partial_1^2 \Gamma(q + te_1 - y) \chi_B(y) \, dy \\ &= |B| \partial_1^2 \Gamma(q - x_B + te_1) + O(t^{-n-1}) \end{aligned}$$

for some fixed $x_B \in \mathbb{R}^n$. For t sufficiently large, $\partial_1 \Phi_B(p - x_B + te_1) = 0$ thus implies $|B| = 0$. This clearly contradicts our assumption that B is open and nonempty. Hence Ω must be connected. \square

We turn to the complement of a regular local minimizer.

Lemma 3.7.2. *Let $2 \leq n \leq 7$ and let $\Omega \in \mathcal{C}_V$ be a regular local minimizer of \mathcal{E} . Then $\mathbb{R}^n \setminus \overline{\Omega}$ is connected.*

Proof. We first note that for all $F \in \mathcal{C}_V$, the set $\mathbb{R}^n \setminus F$ is also of finite perimeter with $D\chi_{\mathbb{R}^n \setminus F} = -D\chi_F$ and hence $\Phi_{\mathbb{R}^n \setminus F} = -\Phi_F$. Thus, the domain of \mathcal{E} extends naturally to

$$\mathcal{C}_V^c := \{F : \mathbb{R}^n \setminus F \in \mathcal{C}_V\}$$

with $\mathcal{E}(\mathbb{R}^n \setminus F) = \mathcal{E}(F)$ for all $F \in \mathcal{C}_V$. Let Ω be a regular local minimizer. Then $G := \mathbb{R}^n \setminus \overline{\Omega}$ is open with smooth boundary and a local minimizer of \mathcal{E} on \mathcal{C}_V^c . Thus, arguments similar to those used in Lemma 3.7.1 (for G instead of Ω) allow to conclude that G is connected. We indicate only some minor changes: Whereas Ω is bounded, G is not. However, since $\Omega \subset B_R$ for some $R > 0$ any partition $G = A \dot{\cup} B$ for some disjoint open sets A, B must contain the connected set $\mathbb{R}^n \setminus \overline{B}_R$. Thus we may assume without loss of generality that $\mathbb{R}^n \setminus \overline{B}_R \subset A$ and B is bounded. The point p in Step 2 will then necessarily lie in ∂A . \square

Connectedness of the boundary $\partial\Omega$ of a regular local minimizer is obtained by the following topological Lemma.

Lemma 3.7.3. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set that equals the interior of its closure $\text{int}(\overline{\Omega}) = \Omega$. Let Ω and $\mathbb{R}^n \setminus \overline{\Omega}$ be connected. Then $\partial\Omega$ is connected.*

A proof is provided in the appendix for the convenience of the reader.

For $n = 2$ our result can be strengthened: The Jordan-Schoenflies Theorem asserts that local minimizers are homeomorphic to a ball.

Lemma 3.7.4. *Let $n = 2$ and Ω be the open representative of a local minimizer of \mathcal{E} from Theorem 3.2.3. Then Ω is homeomorphic to the ball $B_1 \subset \mathbb{R}^2$.*

Proof. Theorem 3.2.3, Proposition 3.7.1 and Lemma 3.7.3 imply that $\partial\Omega$ is a compact, connected 1-manifold without boundary and hence homeomorphic to \mathbb{S}^1 (see, e.g., [34]). In particular, $\partial\Omega$ is the image of an injective continuous map $\gamma : \mathbb{S}^1 \rightarrow \partial\Omega$. Then the Jordan-Schoenflies Theorem (see, e.g., [14]) implies that there is an homeomorphism $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

$$\varphi(\Omega) = B_1, \quad \varphi(\partial\Omega) = \mathbb{S}^1 \quad \text{and} \quad \varphi(\mathbb{R}^2 \setminus \overline{\Omega}) = \mathbb{R}^2 \setminus \overline{B}_1.$$

\square

3.8 Scaling of the minimal energy

In this section, we prove Theorem 3.2.8. Throughout the whole section, we focus on the three dimensional case ($n=3$). We recall that

$$e(V) = \inf_{F \in \mathcal{C}_V} \mathcal{E}(F)$$

denotes the minimal energy as a function of the prescribed volume V .

3.8.1 Upper bound

To prove the upper bound in Theorem 3.2.8, it is sufficient to find an admissible configuration with sufficiently low energy.

Proof of the upper bound in Theorem 3.2.8. We split the proof into the cases $V \leq 2$ and $V > 2$ (the value 2, however, is inessential). For small volumes $V \leq 2$, the optimal scaling is achieved by balls. Indeed, let B denote a ball of volume 1. Then $V^{\frac{1}{3}}B$ has volume V and we get

$$\mathcal{E}(V^{\frac{1}{3}}B) = \mathcal{P}(B)V^{\frac{2}{3}} + \mathcal{N}(B)V \lesssim V^{\frac{2}{3}} \quad \text{for } V \leq 2. \quad (3.69)$$

We turn to the remaining case $V > 2$. For $L > R > 0$ let Ω be the prolate spheroid

$$\Omega := \left\{ x \in \mathbb{R}^3 : \frac{x_1^2}{L^2} + \frac{x_2^2}{R^2} + \frac{x_3^2}{R^2} \leq 1 \right\}.$$

When $L > R$, the surface area of Ω is given by (see, e.g., [96, p. 214])

$$\mathcal{P}(\Omega) = 2\pi R^2 + \frac{2\pi RL^2}{\sqrt{L^2 - R^2}} \sin^{-1} \left(\frac{\sqrt{L^2 - R^2}}{L} \right) \lesssim RL. \quad (3.70)$$

The nonlocal energy for the spheroid is well-known in the physics literature since the work of Maxwell [68, p.69] (see also [90]). His calculations also show (and exploit) the remarkable property of ellipsoids that their stray field $-\nabla\Phi_\Omega$ is constant in Ω . For $L > R$, the nonlocal energy of Ω is given by (see [44, Eq. (3.23a)])

$$\mathcal{N}(\Omega) = \frac{4\pi R^4 L}{3(L^2 - R^2)} \left(\frac{L}{\sqrt{L^2 - R^2}} \sinh^{-1} \left(\frac{\sqrt{L^2 - R^2}}{R} \right) - 1 \right).$$

Under the assumption $L \geq 2R$, this expression simplifies to

$$\mathcal{N}(\Omega) \sim \frac{R^4}{L} \log \left(\frac{L}{R} \right). \quad (3.71)$$

We now choose $L := \frac{3}{4\pi} \frac{V}{R^2}$ to ensure $|\Omega| = V$. Moreover, we set $R := cV^{\frac{2}{7}}(\log V)^{-\frac{1}{7}}$ where $c > 0$ is the largest number such that

$$\frac{L}{R} = \frac{3}{4\pi} \frac{V^{\frac{1}{7}}}{c^3(\log V)^{\frac{3}{7}}} \geq 2 \quad \text{for all } V \geq 2.$$

With these choices, we have

$$R \sim V^{\frac{2}{7}}(\log V)^{-\frac{1}{7}} \quad \text{and} \quad L \sim V^{\frac{3}{7}}(\log V)^{\frac{2}{7}}.$$

Hence, estimates (3.70) and (3.71) yield

$$\mathcal{E}(\Omega) \lesssim V^{\frac{5}{7}}(\log V)^{\frac{1}{7}} \quad \text{for all } V \geq 2. \quad (3.72)$$

Together, (3.69) and (3.72) yield the upper bound in Theorem 3.2.8. \square

3.8.2 Lower bound

Whereas the lower bound in Theorem 3.2.8 for small V follows directly from the isoperimetric inequality, the case of large V is more involved. In order to prove the latter case, we establish the following interpolation result.

Proposition 3.8.1. *There is $C > 0$ such that for all measurable $\Omega \subset \mathbb{R}^3$ with $|\Omega| \in (0, \infty)$ and finite perimeter, the scale invariant quantities*

$$\widehat{\mathcal{P}}(\Omega) := \frac{\mathcal{P}(\Omega)}{|\Omega|^{\frac{2}{3}}} \quad \text{and} \quad \widehat{\mathcal{N}}(\Omega) := \frac{\mathcal{N}(\Omega)}{|\Omega|}$$

satisfy

$$\frac{\widehat{\mathcal{N}}(\Omega)\widehat{\mathcal{P}}(\Omega)^6}{\log \widehat{\mathcal{P}}(\Omega)} \geq C. \quad (3.73)$$

Note that the isoperimetric inequality assures that $\widehat{\mathcal{P}}(\Omega) \geq \sqrt[3]{36\pi} > 1$. Moreover, our construction for the upper bound of \mathcal{E} provides a sequence $(\Omega_k)_{k \in \mathbb{N}}$ of sets $\Omega_k \subset \mathbb{R}^3$ with $\widehat{\mathcal{N}}(\Omega_k) \rightarrow 0$ such that the left hand side in (3.73) remains bounded.

Before we prove Proposition 3.8.1, we first show that it implies the remaining lower bound for the energy.

Proof of the lower bound in Theorem 3.2.8. The lower bound for small V follows from the isoperimetric inequality in three dimensions

$$\mathcal{E}(\Omega) \geq \mathcal{P}(\Omega) \gtrsim V^{\frac{2}{3}} \quad \text{for all } \Omega \in \mathcal{C}_V.$$

Turning to large V , we apply Young's inequality and (3.73) to get

$$\mathcal{E}(\Omega) \gtrsim \mathcal{P}(\Omega)^{\frac{6}{7}} \mathcal{N}(\Omega)^{\frac{1}{7}} \stackrel{(3.73)}{\gtrsim} |\Omega|^{\frac{5}{7}} \left(\log \widehat{\mathcal{P}}(\Omega) \right)^{\frac{1}{7}} \quad \text{for all } \Omega \in \mathcal{C}_V. \quad (3.74)$$

For the lower bound, it is sufficient to consider only those $\Omega \in \mathcal{C}_V$ which satisfy $\mathcal{N}(\Omega) \leq e(V)$. Since the isoperimetric inequality and (3.73) imply $\widehat{\mathcal{P}}(\Omega)^6 \widehat{\mathcal{N}}(\Omega) \gtrsim 1$, our upper bound on $e(V)$ from the previous section yields

$$\widehat{\mathcal{P}}(\Omega)^6 \gtrsim \frac{1}{\widehat{\mathcal{N}}(\Omega)} \gtrsim \frac{V}{e(V)} \gtrsim V^{\frac{1}{7}} \quad (3.75)$$

for all $V \geq 2$ and all $\Omega \in \mathcal{C}_V$ with $\mathcal{N}(\Omega) \leq e(V)$. Inserting (3.75) into (3.74) yields the lower bound for $V \geq 2$. \square

Our argument for (3.73) is based on ideas and techniques developed for a related problem for superconductors in [17], which have been applied to the micromagnetic setting in [56].

A key ingredient in the proof is an estimate for the so called transition energy, i.e. a lower bound for the energy of Ω in terms of its restriction on $\Omega \cap (\{a\} \times \mathbb{R}^2)$ and $\Omega \cap (\{b\} \times \mathbb{R}^2)$ (see also [56, Lemma 3.4], [17, Lemma 2.2]).

Lemma 3.8.2 (Transition energy). *Let $\Omega \in \mathcal{C}_V$. For almost every $a, b \in \mathbb{R}$ and every $\psi \in H^1(\mathbb{R}^2)$ we have*

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \chi_{\Omega}(b, \cdot) \psi - \chi_{\Omega}(a, \cdot) \psi \, dx' \right| &\leq \mathcal{N}(\Omega)^{\frac{1}{2}} |b - a|^{\frac{1}{2}} \|\nabla' \psi\|_{L^2(\mathbb{R}^2)} \\ &+ \left| \int_{\mathbb{R}^2} \partial_1 \Phi_{\Omega}(b, \cdot) \psi - \partial_1 \Phi_{\Omega}(a, \cdot) \psi \, dx' \right|. \end{aligned} \quad (3.76)$$

Proof. We use an approximation argument where we replace χ_{Ω} by $u \in C_c^{\infty}(\mathbb{R}^3)$ and then consider the limit $u \rightarrow \chi_{\Omega}$ in $L^2(\mathbb{R}^3)$. Let $a, b \in \mathbb{R}$, and φ be the distributional solution of $\Delta \varphi = \partial_1 u$. Applying the fundamental theorem of

calculus, inserting the equation for φ and integrating by parts, we get

$$\begin{aligned}
\int_{\mathbb{R}^2} u(b, \cdot) \psi - u(a, \cdot) \psi \, dx' &= \int_a^b \partial_1 \left(\int_{\mathbb{R}^2} u(x_1, \cdot) \psi \, dx' \right) dx_1 \\
&= \int_a^b \left(\int_{\mathbb{R}^2} \partial_1 u(x_1, \cdot) \psi \, dx' \right) dx_1 = \int_{(a,b) \times \mathbb{R}^2} (\Delta \varphi) \psi \, dx \\
&= - \int_{(a,b) \times \mathbb{R}^2} \nabla' \varphi \cdot \nabla' \psi \, dx + \int_{\mathbb{R}^2} \partial_1 \varphi(b, \cdot) \psi - \partial_1 \varphi(a, \cdot) \psi \, dx'.
\end{aligned} \tag{3.77}$$

By Cauchy-Schwarz and Fubini's theorem applied to the first integral in the last line of (3.77) we get

$$\begin{aligned}
\left| \int_{(a,b) \times \mathbb{R}^2} \nabla' \varphi \cdot \nabla' \psi \, dx \right| &\leq \|\nabla' \varphi\|_{L^2((a,b) \times \mathbb{R}^2)} \|\nabla' \psi\|_{L^2((a,b) \times \mathbb{R}^2)} \\
&\leq \|\nabla \varphi\|_{L^2(\mathbb{R}^3)} |b - a|^{\frac{1}{2}} \|\nabla' \psi\|_{L^2(\mathbb{R}^2)}.
\end{aligned} \tag{3.78}$$

We insert (3.78) into (3.77) and consider the limit $u \rightarrow \chi_\Omega$ in $L^2(\mathbb{R}^3)$. L^2 -theory for the potential implies $\nabla \varphi \rightarrow \nabla \Phi_\Omega$ in $L^2(\mathbb{R}^3)$. Upon passing to a subsequence, Fubini's Theorem yields $u(x_1, \cdot) \rightarrow \chi_\Omega(x_1, \cdot)$ and $\nabla \varphi(x_1, \cdot) \rightarrow \nabla \Phi_\Omega(x_1, \cdot)$ in $L^2(\mathbb{R}^2)$ for almost every $x_1 \in \mathbb{R}$. Hence, for almost every $a, b \in \mathbb{R}$, we obtain (3.76) which concludes the proof of the lemma. \square

Before we begin with the proof of Proposition 3.8.1, we record the following approximation Lemma due to De Giorgi [24, Lemma II] (see also [23, Lemma 2.1]).

Lemma 3.8.3. *Let $S \subset \mathbb{R}^2$ be a set of finite perimeter with $|S| < \infty$ and let $r > 0$ satisfy $r\mathcal{P}(S) \leq |S|$. Then there exists an open set $S_r \subset \mathbb{R}^2$ with the properties*

$$(i) \quad |S \cap S_r| \geq \frac{1}{2} |S|.$$

(ii) *For all $t > 0$ the set $S_r^t := \{x \in \mathbb{R}^n \mid \text{dist}(x, S_r) < t\}$ satisfies*

$$|S_r^t| \leq C |S| \left(1 + \left(\frac{t}{r} \right)^n \right)$$

where C is a universal constant.

For a proof of Lemma 3.8.3 we refer to [24, Lemma II] or [23, Lemma 2.1].

Proof of Proposition 3.8.1. Let $\Omega \in \mathcal{C}_V$. By an approximation argument, we may assume that Ω is bounded and has smooth boundary. For $h \in \mathbb{R}$, we call the set

$$S(h) := \{x' \in \mathbb{R}^2 \mid (h, x') \in \Omega\}$$

the slice of Ω at h . Its \mathcal{H}^2 measure is denoted by $|S(h)|$. We note that since Ω has smooth boundary, we have $|S(h)| \leq \mathcal{P}(\Omega)$ for all $h \in \mathbb{R}$.

The argument is based on Lemma 3.8.2 and divided into four steps.

Step 1: Exclude slices with above-average stray field energy. We want to apply the transition energy estimate (3.76) for suitable values $a, b \in \mathbb{R}$ such that the first term on the right hand side of (3.76) dominates the terms involving $\partial_1 \Phi_\Omega(x_1, \cdot)$ for $x_1 \in \{a, b\}$. To this end, let $H \subset \mathbb{R}$ be a measurable set of size

$$|H| = \frac{V}{2\mathcal{P}(\Omega)} \quad (3.79)$$

such that for all $h \in H$ and all $\ell \in \mathbb{R} \setminus H$, we have (see Lemma B.4 for details)

$$\int_{\mathbb{R}^2} |\nabla \Phi_\Omega(\ell, x')|^2 dx' \leq \int_{\mathbb{R}^2} |\nabla \Phi_\Omega(h, x')|^2 dx'$$

and hence

$$\int_{\mathbb{R}^2} |\nabla \Phi_\Omega(\ell, x')|^2 dx' \leq \int_H \int_{\mathbb{R}^2} |\nabla \Phi_\Omega(x_1, x')|^2 dx' dx_1 \leq \frac{\mathcal{N}(\Omega)}{|H|}. \quad (3.80)$$

Our choice (3.79) is such that at least half of Ω 's volume is in $(\mathbb{R} \setminus H) \times \mathbb{R}^2$. Indeed, using $|S(h)| \leq \mathcal{P}(\Omega)$ for $h \in H$ and inserting (3.79), we obtain

$$\begin{aligned} \int_{\mathbb{R} \setminus H} |S(x_1)| dx_1 &= \int_{\mathbb{R}} |S(x_1)| dx_1 - \int_H |S(x_1)| dx_1 \\ &\geq V - |H| \mathcal{P}(\Omega) \stackrel{(3.79)}{\geq} \frac{V}{2}. \end{aligned} \quad (3.81)$$

Furthermore, extending H by a set of measure zero, we may also assume that Lemma 3.8.2 holds for all $x_1 \in \mathbb{R} \setminus H$.

Step 2: Identification of two suitable slices. We identify suitable $a, b \in \mathbb{R} \setminus H$ in order to apply Lemma 3.8.2 for these values a, b .

As a proxy for the radius of Ω we introduce the quantity

$$r := \sup_{x_1 \in \mathbb{R} \setminus H} \frac{|S(x_1)|}{\mathcal{H}^1(\partial S(x_1))}. \quad (3.82)$$

The isoperimetric inequality in two dimensions states that $\mathcal{H}^1(\partial S)^2 \geq 4\pi|S|$ and implies that r is finite

$$r = \sup_{x_1 \in \mathbb{R} \setminus H} \frac{|S(x_1)|}{\mathcal{H}^1(\partial S(x_1))} \leq \frac{1}{2\sqrt{\pi}} \sup_{x_1 \in \mathbb{R} \setminus H} |S(x_1)|^{\frac{1}{2}} \lesssim \mathcal{P}(\Omega)^{\frac{1}{2}}. \quad (3.83)$$

Furthermore, we can estimate r from below in terms of the volume and the interfacial energy

$$\frac{V}{2} \stackrel{(3.81)}{\leq} \int_{\mathbb{R} \setminus H} |S(x_1)| dx_1 \stackrel{(3.82)}{\leq} r \int_{\{|S|>0\} \setminus H} \mathcal{H}^1(\partial S(x_1)) dx_1 \leq r \mathcal{P}(\Omega). \quad (3.84)$$

Now we take $a \in \mathbb{R} \setminus H$ such that (cf. (3.82))

$$\frac{|S(a)|}{\mathcal{H}^1(\partial S(a))} \geq \frac{r}{2}. \quad (3.85)$$

$S(a)$ is the first of the two slices that we will use for the transition energy estimate. In view of the isoperimetric inequality, we have

$$|S(a)| \gtrsim \frac{|S(a)|^2}{\mathcal{H}^1(\partial S(a))^2} \stackrel{(3.85)}{\geq} \frac{r^2}{4}. \quad (3.86)$$

It remains to identify the second slice $S(b)$ which should have significantly less area than $S(a)$, lie outside of H and with b close to a . We claim that there is

$$b \in \left(a - \frac{32}{r^2}V, a + \frac{32}{r^2}V\right) \cap (\mathbb{R} \setminus H) \quad \text{with} \quad |S(b)| \leq \frac{1}{4}|S(a)|. \quad (3.87)$$

Indeed, assume for contradiction that no such b exists and thus $|S(x_1)| > \frac{1}{4}|S(a)|$ for all $x_1 \in \left(a - \frac{32}{r^2}V, a + \frac{32}{r^2}V\right) \cap (\mathbb{R} \setminus H)$. Since $|H| = \frac{V}{2\mathcal{P}(\Omega)} \stackrel{(3.83)}{\leq} \frac{V}{2r^2}$ we obtain a contradiction

$$\begin{aligned} V &\geq \int_{\left(a - \frac{32}{r^2}V, a + \frac{32}{r^2}V\right) \cap (\mathbb{R} \setminus H)} |S(x_1)| dx_1 \\ &> \frac{1}{4}|S(a)| \left(V \frac{64}{r^2} - |H|\right) \geq |S(a)| V \frac{8}{r^2} \stackrel{(3.86)}{\geq} 2V. \end{aligned}$$

Hence, we conclude that such a b exists and use $S(b)$ as the second slice.

Step 3: Definition of a suitable testfunction. The construction of the test-function closely follows [17]. For $\lambda \geq 2$, we define the function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\varphi(t) = \begin{cases} 1 & \text{for } t \leq r, \\ \frac{\log(\lambda r/t)}{\log(\lambda)} & \text{for } r \leq t \leq \lambda r, \\ 0 & \text{for } \lambda r \leq t. \end{cases}$$

In view of (3.85), we apply Lemma 3.8.3 to $S(a)$ and r which yields the regularized set S_r . We define the test function $\psi \in H^1(\mathbb{R}^2)$ by $\psi(x) := \varphi(\text{dist}(x, S_r))$. Arguing as in [17], one can show that

$$\int_{\mathbb{R}^2} \psi^2 dx \lesssim \frac{|S(a)|\lambda^2}{(\log \lambda)^2} \quad \text{and} \quad \int_{\mathbb{R}^2} |\nabla' \psi|^2 \lesssim \frac{|S(a)|}{r^2 \log(\lambda)}. \quad (3.88)$$

For the convenience of the reader, we sketch the argument for the first item in (3.88) (the second one is analogous). We use the coarea formula and integrate by parts to obtain

$$\int_{\mathbb{R}^2} \psi^2 dx = |S_r| + \int_0^\infty \varphi^2(t) \mathcal{H}^1(\partial S_r^t) dt = - \int_r^{\lambda r} 2\varphi(t)\varphi'(t)|S_r^t| dt. \quad (3.89)$$

Since $\varphi'(t) = -\frac{1}{t \log(\lambda)}$ on $(r, \lambda r)$, using properties (i) and (ii) from Lemma 3.8.3 turns (3.89) into

$$\begin{aligned} \int_{\mathbb{R}^2} \psi^2 dx &\stackrel{(3.89)}{\lesssim} - \int_r^{\lambda r} \varphi(t)\varphi'(t)|S(a)| \left(\frac{t}{r}\right)^2 dt \\ &\sim |S(a)| \int_r^{\lambda r} \frac{t \log(\lambda r/t)}{r^2 (\log \lambda)^2} dt \sim \frac{|S(a)|\lambda^2}{(\log \lambda)^2}. \end{aligned}$$

Step 4: Derivation of the lower bound. We use (3.76) for a, b and ψ as defined in Steps 2 and 3, respectively. Also applying Hölder's inequality and (3.80), we obtain

$$\begin{aligned} &\left| \int_{\mathbb{R}^2} \chi_\Omega(b, \cdot) \psi - \chi_\Omega(a, \cdot) \psi dx' \right| \\ &\stackrel{(3.76)}{\leq} \mathcal{N}(\Omega)^{\frac{1}{2}} |b - a|^{\frac{1}{2}} \|\nabla' \psi\|_{L^2(\mathbb{R}^2)} \\ &\quad + \left| \int_{\mathbb{R}^2} \partial_1 \Phi_\Omega(b, \cdot) \psi - \partial_1 \Phi_\Omega(a, \cdot) \psi dx' \right| \\ &\stackrel{(3.80)}{\leq} \mathcal{N}(\Omega)^{\frac{1}{2}} \left(|b - a| \|\nabla' \psi\|_{L^2(\mathbb{R}^2)}^2 + \frac{2\mathcal{P}(\Omega)}{V} \|\psi\|_{L^2(\mathbb{R}^2)}^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (3.90)$$

The left hand side of (3.90) may be bounded below using Lemma 3.8.3 (i) and (3.87)

$$\begin{aligned} \int_{\mathbb{R}^2} (\chi_{\Omega}(a, \cdot) - \chi_{\Omega}(b, \cdot)) \psi \, dx' &\geq |S(a) \cap S_r| - |S(b)| \\ &\geq \frac{1}{2} |S(a)| - \frac{1}{4} |S(a)| \gtrsim |S(a)|. \end{aligned} \quad (3.91)$$

Together, (3.91) and (3.90) imply

$$|S(a)|^2 \lesssim \mathcal{N}(\Omega) \left(|b - a| \|\nabla \psi\|_{L^2(\mathbb{R}^2)}^2 + \frac{2\mathcal{P}(\Omega)}{V} \|\psi\|_{L^2(\mathbb{R}^2)}^2 \right). \quad (3.92)$$

Inserting (3.88) into (3.92) and dividing by $|S(a)|$, we get

$$|S(a)| \lesssim \mathcal{N}(\Omega) \left(\frac{|b - a|}{r^2 \log \lambda} + \frac{2\mathcal{P}(\Omega)\lambda^2}{V(\log \lambda)^2} \right). \quad (3.93)$$

In turn, inserting (3.86) and the first item in (3.87) into (3.93), we obtain

$$r^2 \stackrel{(3.86)}{\lesssim} |S(a)| \lesssim \mathcal{N}(\Omega) \left(\frac{V}{r^4 \log \lambda} + \frac{2\mathcal{P}(\Omega)\lambda^2}{V(\log \lambda)^2} \right). \quad (3.94)$$

Finally, inserting (3.84) into (3.94), we arrive at

$$\left(\frac{V}{\mathcal{P}(\Omega)} \right)^2 \lesssim \mathcal{N}(\Omega) \left(\frac{\mathcal{P}(\Omega)^4}{V^3 \log \lambda} + \frac{2\mathcal{P}(\Omega)\lambda^2}{V(\log \lambda)^2} \right). \quad (3.95)$$

The isoperimetric inequality asserts that the choice $\lambda = \frac{\mathcal{P}(\Omega)^{\frac{3}{2}}}{V} \geq 6\sqrt{\pi}$ is admissible. Inserting it into (3.95) yields

$$V^5 \log \left(\frac{\mathcal{P}(\Omega)^{\frac{3}{2}}}{V} \right) \leq C \mathcal{N}(\Omega) \mathcal{P}(\Omega)^6$$

which is equivalent to (3.73). □

Appendix

Appendix A

Here we provide supplementary material to chapter 1. In particular, we give a proof for the continuity of $\varepsilon \mapsto \lambda_c(\varepsilon)$, the critical value of λ (see also Figure 1.3). Furthermore, we record a few well-known results that are used throughout chapter 1. For the convenience of the reader, we also give the proofs.

For $0 < \varepsilon < 1$, we define the critical value of λ where $\min F_{\varepsilon,\lambda}$ becomes negative as

$$\lambda_c(\varepsilon) := \inf\{\lambda : \min F_{\varepsilon,\lambda} < 0\}. \quad (\text{A.1})$$

Lemma A.1. *The function $\lambda_c : (0, 1) \rightarrow \mathbb{R}$ (see (A.1)) is Lipschitz-continuous on compact subsets of $(0, 1)$.*

Proof. The main idea is to express λ_c as the infimum over $\lambda_{c,m}$, where m is held fixed (see (A.2)) and to deduce regularity of λ_c from the regularity of $\lambda_{c,m}$. We define

$$X := \{m \in H^1(\mathbb{T}^2; \mathbb{S}^2) : m \text{ is not constant}\}$$

and introduce, for any $m \in X$, the function

$$\lambda_{c,m} : (0, 1) \rightarrow \mathbb{R}, \quad \varepsilon \mapsto \lambda_{c,m}(\varepsilon) := \inf\{\lambda : F_{\varepsilon,\lambda}[m] < 0\}. \quad (\text{A.2})$$

Note that $F_{\varepsilon,\lambda}[m] \geq 0$ if m is constant and that $\lambda \mapsto F_{\varepsilon,\lambda}[m]$ is strictly monotone (for ε and $m \in X$ fixed). Hence, we may rewrite

$$\begin{aligned} \lambda_c(\varepsilon) &= \inf\{\lambda : \exists m \in X \text{ s.t. } F_{\varepsilon,\lambda}[m] < 0\} \\ &= \inf\{\lambda : \exists m \in X \text{ s.t. } \lambda > \lambda_{c,m}(\varepsilon)\} = \inf_{m \in X} \lambda_{c,m}(\varepsilon). \end{aligned} \quad (\text{A.3})$$

Step 1: Regularity of $\lambda_{c,m}$. We claim that

$$\left| \frac{d}{d\varepsilon} \lambda_{c,m}(\varepsilon) \right| \leq \left(1 + \frac{1}{|\log \varepsilon|} \right) \frac{\lambda_{c,m}(\varepsilon)}{\varepsilon} \quad \text{for all } m \in X. \quad (\text{A.4})$$

To prove (A.4), fix $m \in X$ and abbreviate

$$a = \int_{\mathbb{T}^2} |\nabla m|^2 dx, \quad b := \int_{\mathbb{T}^2} (1 - m_1^2) dx \quad c := \int_{\mathbb{T}^2} |\nabla^{\frac{1}{2}} m_1|^2 dx,$$

so that $F_{\varepsilon,\lambda}[m] = \frac{\varepsilon}{2}a + \frac{b}{2\varepsilon} - \frac{\lambda}{|\log \varepsilon|}c$ with partial derivatives

$$\partial_\varepsilon F_{\varepsilon,\lambda}[m] = \frac{a}{2} - \frac{b}{2\varepsilon^2} - \frac{\lambda c}{\varepsilon |\log \varepsilon|^2} \quad \text{and} \quad \partial_\lambda F_{\varepsilon,\lambda}[m] = -\frac{c}{|\log \varepsilon|}.$$

By continuity of $(\varepsilon, \lambda) \mapsto F_{\varepsilon,\lambda}[m]$ and strict monotonicity in λ , we deduce from (A.2) that $\lambda_{c,m}$ satisfies $F_{\varepsilon,\lambda_{c,m}(\varepsilon)}[m] = 0$ for all $\varepsilon \in (0, 1)$ and, furthermore, that it is the only function with this property. Then the implicit function theorem asserts that $\lambda_{c,m}$ is $C^1((0, 1))$ with

$$\begin{aligned} \frac{d}{d\varepsilon} \lambda_{c,m}(\varepsilon) &= -(\partial_\lambda F_{\varepsilon,\lambda}[m])^{-1} \partial_\varepsilon F_{\varepsilon,\lambda}[m] \\ &= \frac{|\log \varepsilon|}{c} \left(\frac{a}{2} - \frac{b}{2\varepsilon^2} - \frac{\lambda c}{\varepsilon |\log \varepsilon|^2} \right). \end{aligned} \quad (\text{A.5})$$

Inserting the identity $F_{\varepsilon,\lambda_{c,m}(\varepsilon)}[m] = \frac{\varepsilon}{2}a + \frac{b}{2\varepsilon} - \frac{\lambda_{c,m}(\varepsilon)}{|\log \varepsilon|}c = 0$ into (A.5), we obtain the estimate

$$\left| \frac{d}{d\varepsilon} \lambda_{c,m}(\varepsilon) \right| \leq \frac{|\log \varepsilon|}{\varepsilon} \left(\frac{\frac{\varepsilon a}{2} + \frac{b}{2\varepsilon}}{c} \right) + \frac{\lambda_{c,m}(\varepsilon)}{\varepsilon |\log \varepsilon|} \leq \left(1 + \frac{1}{|\log \varepsilon|} \right) \frac{\lambda_{c,m}(\varepsilon)}{\varepsilon}$$

which completes the proof of (A.4).

Step 2: Regularity of λ_c . The metric space $(X, \|\cdot\|_{H^1})$ is separable as a subset of the separable metric space $H^1(\mathbb{T}^2; \mathbb{R}^3)$ and hence there exists a dense countable subset $\{m_n : n \in \mathbb{N}\} \subset X$. Let $\delta \in (0, 1/2)$ and define $M := \sup_{\varepsilon \in [\delta, 1-\delta]} |\lambda_{c,m_1}(\varepsilon)| < +\infty$. Then the functions

$$g_n : [\delta, 1-\delta] \rightarrow \mathbb{R}, \quad \varepsilon \mapsto g_n(\varepsilon) = \min\{\lambda_{c,m_n}(\varepsilon), M\}$$

are Lipschitz-continuous for all $n \in \mathbb{N}$. Furthermore, by (A.4), their Lipschitz-constant is bounded by $\delta^{-1}(1 + \frac{1}{|\log \delta|})M$ (independent of $n \in \mathbb{N}$). Define the sequence of functions $f_k := \min_{1 \leq n \leq k} g_n$ and observe that

$$(i) \quad \|f_k\|_{C^0([\delta, 1-\delta])} \leq M \text{ for all } k \in \mathbb{N},$$

- (ii) f_k is Lipschitz continuous with Lipschitz constant bounded by $\delta^{-1}(1 + \frac{1}{|\log \delta|})M$,
- (iii) $f_k(\varepsilon) \rightarrow \lambda_c(\varepsilon)$ as $k \rightarrow \infty$ for all $\varepsilon \in [\delta, 1 - \delta]$.

The last point follows from (A.3), the density of $\{m_n : n \in \mathbb{N}\} \subset X$ and continuity of $m \mapsto F_{\varepsilon, \lambda}[m]$. Now the compact embedding $C^{0,1}([\delta, 1 - \delta]) \hookrightarrow C^0([\delta, 1 - \delta])$ implies that $f_k \rightarrow f$ uniformly for some $f \in C^{0,1}([\delta, 1 - \delta])$ with Lipschitz constant bounded by $\delta^{-1}(1 + \frac{1}{|\log \delta|})M$. By uniqueness of the limit we conclude that $f = \lambda_c$, which completes the proof. \square

It is well-known that if $m \in H^1$ takes values in \mathbb{S}^2 , this implies certain estimates for the gradient ∇m (see, e.g., [59]). Since these estimates are used frequently throughout this thesis, we record them in the following Lemma.

Lemma A.2. *Let $\Omega \subset \mathbb{R}^n$ be open and $m \in H^1(\Omega, \mathbb{S}^2)$. Then for every $\varepsilon > 0$ we have*

$$(i) \quad \frac{|\nabla m_1|^2}{1 - m_1^2} \leq |\nabla m|^2 \quad \text{for a.e. } x \in \Omega \text{ with } |m_1(x)| < 1, \quad (\text{A.6})$$

$$(ii) \quad |\nabla m_1| \leq \frac{\varepsilon}{2} |\nabla m|^2 + \frac{1 - m_1^2}{2\varepsilon} \quad \text{for a.e. } x \in \Omega. \quad (\text{A.7})$$

Proof. To prove (i), we apply the weak product rule to the constraint $|m|^2 = 1$, which yields

$$-m_1 \nabla m_1 = m_2 \nabla m_2 + m_3 \nabla m_3$$

a.e. in Ω . After squaring both sides and applying the n -dimensional Cauchy-Schwarz inequality, we obtain

$$m_1^2 |\nabla m_1|^2 \leq (m_2^2 + m_3^2) (|\nabla m_2|^2 + |\nabla m_3|^2).$$

Finally we add $(m_2^2 + m_3^2) |\nabla m_1|^2$ to both sides. Since $|m|^2 = 1$, this yields

$$|\nabla m_1|^2 \leq (1 - m_1^2) |\nabla m|^2,$$

and hence proves (A.6).

We turn to the proof of (ii). Since $\nabla m_1 = 0$ almost everywhere on the set $\{x \in \Omega : |m_1(x)| = 1\}$, it remains to prove (A.7) on $\{x \in \Omega, |m_1(x)| < 1\}$. This follows from (A.6) upon an application of Young's inequality

$$2|\nabla m_1| \leq \frac{\varepsilon |\nabla m_1|^2}{1 - m_1^2} + \frac{1 - m_1^2}{\varepsilon} \stackrel{(\text{A.6})}{\leq} \varepsilon |\nabla m|^2 + \frac{1 - m_1^2}{\varepsilon},$$

which concluded the proof. \square

In the following Lemma, we record a consequence of Jensen's inequality for the gradients of e_1 -averages.

Lemma A.3. *For every $p \in [1, \infty)$ and every $f \in W^{1,p}((0, 1) \times \mathbb{T}^2)$, we have*

$$\int_{\mathbb{T}^2} \left| \nabla' \int_0^1 f(x_1, x') \, dx_1 \right|^p dx' \leq \int_{(0,1) \times \mathbb{T}^2} |\nabla' f|^p \, dx. \quad (\text{A.8})$$

Proof. Assume for a moment that $f \in C^\infty((0, 1) \times \mathbb{T}^2)$. Since $|\cdot|^p : \mathbb{R}^2 \rightarrow \mathbb{R}$ (the p -th power of the euclidean norm) is a convex function, an application of Jensen's inequality (for two-dimensions) then yields

$$\left| \int_0^1 \nabla' f(x_1, x') \, dx_1 \right|^p dx' \leq \int_0^1 |\nabla' f(x_1, x')|^p \, dx_1 \quad (\text{A.9})$$

for all $x' \in \mathbb{T}^2$. For $f \in C^\infty((0, 1) \times \mathbb{T}^2)$, we can change the order of integration and differentiation, so that (A.8) follows from (A.9) upon integrating over \mathbb{T}^2

$$\begin{aligned} \int_{\mathbb{T}^2} \left| \nabla' \int_0^1 f(x_1, x') \, dx_1 \right|^p dx' &= \int_{\mathbb{T}^2} \left| \int_0^1 \nabla' f(x_1, x') \, dx_1 \right|^p dx' \\ &\stackrel{(\text{A.9})}{\leq} \int_{\mathbb{T}^2} \int_0^1 |\nabla' f(x_1, x')|^p \, dx_1 \, dx'. \end{aligned} \quad (\text{A.10})$$

Finally, (A.10) extends to any $f \in W^{1,p}((0, 1) \times \mathbb{T}^2)$ by a standard approximation argument using lower semi-continuity of the $W^{1,p}(\mathbb{T}^2)$ norm with respect to weak convergence of the e_1 -averages. \square

The next Lemma relates the real space formulation of the homogeneous $\dot{H}^{\frac{1}{2}}$ -norm to its Fourier representation.

Lemma A.4. *For every smooth function $f : \mathbb{T}_\ell^2 \rightarrow \mathbb{R}$, the following holds*

$$\int_{\mathbb{T}_\ell^2} |\nabla^{\frac{1}{2}} f|^2 \, dx := \frac{1}{\ell^2} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} |k| |\widehat{f}_k|^2 = \frac{1}{4\pi} \int_{\mathbb{T}_\ell^2} \int_{\mathbb{R}^2} \frac{|f(x) - f(y)|^2}{|x - y|^3} \, dx \, dy. \quad (\text{A.11})$$

Proof. First we prove the identity

$$\int_{\mathbb{R}^2} |e^{ik \cdot x} - 1|^2 \frac{1}{|x|^3} \, dx = 4\pi |k| \quad \text{for every } k \in \frac{2\pi}{\ell} \mathbb{Z}^2. \quad (\text{A.12})$$

By scaling and rotational symmetry, we have

$$\int_{\mathbb{R}^2} |e^{ik \cdot x} - 1|^2 \frac{1}{|x|^3} dx = |k| \int_{\mathbb{R}^2} |e^{ix_1} - 1|^2 \frac{1}{|x|^3} dx. \quad (\text{A.13})$$

We evaluate the last integral in polar coordinates. On substituting $\rho = \frac{r \cos \theta}{2}$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^2} |e^{ix_1} - 1|^2 \frac{1}{|x|^3} dx &= \int_{\mathbb{R}^2} |e^{\frac{ix_1}{2}} - e^{-\frac{ix_1}{2}}|^2 \frac{1}{|x|^3} dx = \int_0^{2\pi} \int_0^\infty 4 \sin^2 \left(\frac{r \cos \theta}{2} \right) \frac{1}{r^3} r d\theta dr \\ &= 2 \int_0^{2\pi} |\cos \theta| d\theta \int_0^\infty \frac{\sin^2 \rho}{\rho^2} d\rho = 4\pi. \end{aligned} \quad (\text{A.14})$$

Together, (A.13) and (A.14) prove (A.12).

With (A.12) at hand, we will now prove (A.11). By a variable transformation and Fubini's Theorem, we obtain

$$\int_{\mathbb{T}_\ell^2} \int_{\mathbb{R}^2} \frac{|f(x) - f(y)|^2}{|x - y|^3} dx dy = \int_{\mathbb{R}^2} \int_{\mathbb{T}_\ell^2} |f(z + y) - f(y)|^2 dy \frac{1}{|z|^3} dz.$$

Rewriting the inner integral in Fourier space and using Fubini's Theorem again yields

$$\begin{aligned} \int_{\mathbb{R}^2} \int_{\mathbb{T}_\ell^2} |f(z + y) - f(y)|^2 dy \frac{1}{|z|^3} dz &= \frac{1}{\ell^2} \int_{\mathbb{R}^2} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} |e^{-ik \cdot z} - 1|^2 |\widehat{f}_k|^2 \frac{1}{|z|^3} dz \\ &= \frac{1}{\ell^2} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} |\widehat{f}_k|^2 \int_{\mathbb{R}^2} |e^{-ik \cdot z} - 1|^2 \frac{1}{|z'|^3} dz \stackrel{(\text{A.12})}{=} \frac{4\pi}{\ell^2} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} |k| |\widehat{f}_k|^2, \end{aligned}$$

which gives the desired formula. \square

Appendix B

Lemma B.1. *Let $\Omega \subset \mathbb{R}^n$ satisfy $|\Omega| < \infty$ and define $\Psi_\Omega : \mathbb{R}^n \rightarrow \mathbb{R}$ by*

$$\Psi_\Omega(x) = \int_{\mathbb{R}^n} \partial_1 \Gamma(x-y) \chi_\Omega(y) \, dy. \quad (\text{B.1})$$

Then $\Psi_\Omega \in L^1_{loc}(\mathbb{R}^n)$ is a distributional solution to (3.2).

Proof. Step 1: We show that $\Psi_\Omega \in L^1_{loc}(\mathbb{R}^n)$ is a distributional solution of $\Delta \Psi_\Omega = \partial_1 \chi_\Omega$. First note that the integral (B.1) is well-defined for all $x \in \mathbb{R}^n$, because $|\Omega| < \infty$ and the singularity of $\partial_1 \Gamma$ is integrable, i.e. $\partial_1 \Gamma \in L^1_{loc}(\mathbb{R}^n)$. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$. Inserting (B.1), applying Fubini's theorem and integrating by parts, we get

$$\begin{aligned} \int_{\mathbb{R}^n} \Psi_\Omega(x) \Delta \varphi(x) \, dx &\stackrel{(\text{B.1})}{=} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \partial_{x_1} \Gamma(x-y) \Delta \varphi(x) \, dx \right) \chi_\Omega(y) \, dy \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} -\Gamma(y-x) \partial_1 \Delta \varphi(x) \, dx \right) \chi_\Omega(y) \, dy. \end{aligned}$$

Upon inserting the identity $\partial_1 \varphi(y) = (\Gamma * \partial_1 \Delta \varphi)(y)$, we conclude that $\Psi_\Omega \in L^1_{loc}(\mathbb{R}^n)$ is indeed a distributional solution of $\Delta \Psi_\Omega = \partial_1 \chi_\Omega$.

Step 2: We show that Ψ_Ω satisfies the decay condition $\lim_{|x| \rightarrow \infty} \Psi_\Omega(x) = 0$. By monotonicity of $|\partial_1 \Gamma|$, we have

$$\left| \int_{B_{|x|/2}} \partial_1 \Gamma(x-y) \chi_\Omega(y) \, dy \right| \leq \sup_{y \in B_{|x|/2}} |\partial_1 \Gamma(x-y)| |\Omega| = |\partial_1 \Gamma(x/2)| |\Omega|. \quad (\text{B.2})$$

Since $|\partial_1 \Gamma| \in L^1_{loc}(\mathbb{R}^n)$ is spherically monotone, the remaining part may be estimated using rearrangements of $\Omega \setminus B_{|x|/2}$. Let $\tilde{B}(x)$ be the ball of volume $|\Omega \setminus B_{|x|/2}|$, then

$$\left| \int_{\mathbb{R}^n \setminus B_{|x|/2}} \partial_1 \Gamma(x-y) \chi_\Omega(y) \, dy \right| \leq \int_{\tilde{B}(x)} |\partial_1 \Gamma|(y) \, dy. \quad (\text{B.3})$$

Since $|\tilde{B}(x)| = |\Omega \setminus B_{|x|/2}| \rightarrow 0$ for $|x| \rightarrow \infty$ (e.g. by dominated convergence), adding (B.2) and (B.3) yields the claim

$$|\Psi_\Omega(x)| \leq |\partial_1 \Gamma(x/2)| |\Omega| + \int_{\tilde{B}(x)} |\partial_1 \Gamma|(y) \, dy \longrightarrow 0, \text{ for } |x| \rightarrow \infty.$$

□

Lemma B.2. *Let T be given by (3.19) and $f \in C_c^\infty(\mathbb{R}^n)$. Then*

$$\nabla(\partial_1 \Gamma * f) = Tf.$$

Proof. Since $\partial_1 \Gamma \in L_{loc}^1(\mathbb{R}^n)$, we have

$$\partial_k(\partial_1 \Gamma * f)(x) = (\partial_1 \Gamma * \partial_k f)(x) = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \partial_1 \Gamma(y) \partial_k f(x - y) dy \quad (\text{B.4})$$

for all $1 \leq k \leq n$ and every $x \in \mathbb{R}^n$. Moreover, for any $\varepsilon > 0$, an integration by parts yields

$$\begin{aligned} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \partial_1 \Gamma(y) \partial_k f(x - y) dy &= \int_{\partial B_\varepsilon(x)} \partial_1 \Gamma(y) f(x - y) \frac{y_k - x_k}{|y - x|} dy \\ &\quad + \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \partial_{k,1}^2 \Gamma(y) f(x - y) dy. \end{aligned} \quad (\text{B.5})$$

Since f is smooth, we have $f(x - y) = f(x) + O(|y|)$ for $y \rightarrow x$ and in the limit $\varepsilon \rightarrow 0$, the boundary term yields

$$\lim_{\varepsilon \rightarrow 0} \int_{\partial B_\varepsilon(x)} \partial_1 \Gamma(y) f(x - y) \frac{y_k - x_k}{|y - x|} dy = \frac{f(x)}{n\omega_n} \int_{\mathbb{S}^{n-1}} \frac{z_k z_1}{|z|^{n+1}} dz = \frac{\delta_{1,k}}{n} f(x).$$

Together, (B.4) and (B.5) yield the claim

$$\nabla(\partial_1 \Gamma * f) = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n \setminus B_\varepsilon} \partial_{k,1}^2 \Gamma(y) f(\cdot - y) dy + \frac{\delta_{1,k}}{n} f e_1.$$

□

Lemma B.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $C^{1,\alpha}$ -boundary for some $\alpha \in (0, 1]$ and let $\{F_t\}_{|t| < t_0}$ be a local variation for which we set $\Omega_t = F_t(\Omega)$. Let $\rho \in C_c^\infty(\mathbb{R}^n)$ be a rotational symmetric mollification kernel and set $\rho_\varepsilon(x) := \frac{1}{\varepsilon^n} \rho(\frac{x}{\varepsilon})$. Then there is $t_* \in (0, t_0)$ such that for all $t \in [-t_*, t_*]$ and all $x \in \partial\Omega_t$*

$$- \int_{\partial\Omega_t} (\rho_\varepsilon * \nabla \Gamma)(x - y) \nu_1(y) d\mathcal{H}^{n-1}(y) \xrightarrow{\varepsilon \rightarrow 0} \langle \nabla \Phi_{\Omega_t} \rangle(x)$$

uniformly in t and x .

The main point is that due to uniform $C^{1,\alpha}$ -bounds on $\partial\Omega_t$ the convergence is uniform.

Proof of Lemma B.3. We abbreviate $k_\varepsilon = (\rho_\varepsilon * \partial_1 \Gamma) - \partial_1 \Gamma$ and assume without loss of generality that $\text{supp } \rho_\varepsilon \subset B_{\frac{\varepsilon}{2}}$. Since ρ_ε is rotational symmetric and $\partial_1 \Gamma$ is harmonic on $\mathbb{R}^n \setminus \{0\}$ we conclude that $k_\varepsilon \equiv 0$ on $\mathbb{R}^n \setminus B_\varepsilon$. We rewrite

$$\begin{aligned} & -(\rho_\varepsilon * \partial_1 \Phi_{\Omega_t})(x) + \langle \partial_1 \Phi_{\Omega_t} \rangle(x) \\ &= \int_{\partial \Omega_t} (\rho_\varepsilon * \partial_1 \Gamma)(x-y) \nu_1(y) \, dy - \lim_{\delta \rightarrow 0} \int_{\partial \Omega_t \setminus B_\delta(x)} \partial_1 \Gamma(x-y) \nu_1(y) \, dy \\ &= \lim_{\delta \rightarrow 0} \int_{\partial \Omega_t \cap (B_\varepsilon(x) \setminus B_\delta(x))} k_\varepsilon(x-y) \nu_1(y) \, dy \quad \text{for all } x \in \partial \Omega_t. \end{aligned} \quad (\text{B.6})$$

Since the $C^{1,\alpha}$ -boundary $\partial \Omega$ is compact and Ω_t is a smooth deformation of Ω , there are $C, r, > 0$ and $t_* \in (0, t_0)$ with the following properties: For all $t \in (-t_*, t_*)$ and all $x \in \partial \Omega_t$ there is $\gamma_{x,t} \in C^{1,\alpha}(\mathbb{R}^{n-1})$ with $\|\gamma_{x,t}\|_{C^{1,\alpha}} \leq C$ and – up to relabeling and reorienting the coordinate axes if necessary – we have

$$\Omega_t \cap B_r(x) = \{(y_1, y') \in B_r(x) : y_1 - x_1 < +\gamma_{x,t}(y' - x')\}.$$

Moreover, ν is $C^{0,\alpha}$ -continuous on $\partial \Omega_t$ with t -independent bounds. Since $|k_\varepsilon(z)| \lesssim |z|^{1-n}$, we conclude that for all $\delta > 0$ and all $\varepsilon < r$ we have

$$\begin{aligned} & \int_{\partial \Omega_t \cap (B_\varepsilon(x) \setminus B_\delta(x))} k_\varepsilon(x-y) (\nu_1(y) - \nu_1(x)) \, dy \\ & \leq C_\Omega \int_{\partial \Omega_t \cap (B_\varepsilon(x) \setminus B_\delta(x))} \frac{1}{|x-y|^{n-1-\alpha}} \, dy \leq C_\Omega \varepsilon^\alpha \end{aligned} \quad (\text{B.7})$$

for some generic constant C_Ω which depends on Ω but is independent of t, ε and δ . In view of (B.6) and (B.7), it is sufficient to show that

$$\int_{\partial \Omega_t \cap (B_\varepsilon(x) \setminus B_\delta(x))} k_\varepsilon(x-y) \, dy$$

vanishes for $\varepsilon \rightarrow 0$ uniformly in t and δ . To simplify the notation, we assume in the following (without loss of generality) that $x = 0$ and that $\gamma := \gamma_{x,t}$ satisfies $\nabla' \gamma(0) = 0$. The integral over $\partial \Omega_t \cap (B_\varepsilon \setminus B_\delta)$ can be written in the parametrized form

$$\int_{\Omega_t \cap (B_\varepsilon \setminus B_\delta)} k_\varepsilon(y) \, dy = \int_M k_\varepsilon(\gamma(y')e_1 + y') \sqrt{1 + |\nabla' \gamma|^2} \, dy'$$

where $M := \{y' \in B'_r : \delta^2 \leq \gamma^2(y') + |y'|^2 < \varepsilon^2\}$ and B'_r denotes a ball in \mathbb{R}^{n-1} . It is convenient to split $M = R_1 \cup A \cup R_2$ into a maximal annulus

$A := B_{r_2} \setminus B_{r_1} \subset M$ and boundary sets $R_1 = M \cap B_{r_1}$ and $R_2 := M \setminus B_{r_2}$. We begin with the estimate for the integral on the annulus A which we rewrite as

$$\begin{aligned} & \int_A k_\varepsilon(\gamma(y')e_1 + y') \sqrt{1 + |\nabla' \gamma|^2} dy' \\ &= \frac{1}{2} \int_A \left(k_\varepsilon(\gamma(y')e_1 + y') \sqrt{1 + |\nabla' \gamma(y')|^2} \right. \\ & \quad \left. + k_\varepsilon(\gamma(-y')e_1 - y') \sqrt{1 + |\nabla' \gamma(-y')|^2} \right) dy' \end{aligned}$$

in order to exploit cancellation. An elementary estimate using $|\gamma(y')| \lesssim |y'|^{1+\alpha}$ yields

$$|k_\varepsilon(\gamma(y')e_1 + y') + k_\varepsilon(\gamma(-y')e_1 - y')| \lesssim \frac{1}{|y'|^{n-1-\alpha}}.$$

Since $1 \leq \sqrt{1 + |\nabla' \gamma|^2} \leq 1 + C|y'|^\alpha$ and $|k_\varepsilon(y')| \lesssim \frac{1}{|y'|^{n-1}}$ we get

$$\begin{aligned} & \left| k_\varepsilon(\gamma(y')e_1 + y') \sqrt{1 + |\nabla' \gamma(y')|^2} + k_\varepsilon(\gamma(-y')e_1 - y') \sqrt{1 + |\nabla' \gamma(-y')|^2} \right| \\ & \lesssim \frac{1}{|y'|^{n-1}} \end{aligned}$$

and thus

$$\int_A k_\varepsilon(\gamma(y')e_1 + y') \sqrt{1 + |\nabla' \gamma|^2} dy' \lesssim \int_{r_1}^{r_2} \frac{1}{\rho^{1-\alpha}} d\rho \lesssim r_2^\alpha \lesssim \varepsilon^\alpha. \quad (\text{B.8})$$

For the estimate of the boundary set R_2 note that $R_2 \subset B'_\varepsilon \setminus B'_{r_2}$ and that $|\varepsilon - r_2| \leq \varepsilon^{1+2\alpha}$. Thus

$$\int_{R_2} k_\varepsilon(y') \sqrt{1 + |\nabla' \gamma|^2} dy' \lesssim \int_{r_2}^\varepsilon \frac{1}{\rho^{1-\alpha}} d\rho \sim \log\left(\frac{\varepsilon}{r_2}\right) \lesssim \frac{\varepsilon - r_2}{r_2} \sim \varepsilon^{2\alpha}. \quad (\text{B.9})$$

The estimate for R_1 is analogous. Combining (B.8), (B.9) and the analogous estimate on R_1 , we conclude that

$$\int_{\partial\Omega_t \cap (B_\varepsilon(x) \setminus B_\delta(x))} k_\varepsilon(x - y) dy \leq C_\Omega \varepsilon^\alpha \rightarrow 0 \quad (\text{B.10})$$

uniformly in t and δ . Inserting (B.7), (B.10) and $|\nu_1| \leq 1$ into (B.6) yields the claim. \square

For the convenience of the reader, we provide a proof of Lemma 3.7.3. It uses that \mathbb{R}^n is simply connected.

Proof of Lemma 3.7.3. Assume for contradiction that $\partial\Omega$ is not connected and hence can be written as $\partial\Omega = A \cup B$ for some nonempty disjoint sets A, B . Since $\partial\Omega$ is compact, we may assume that A and B are compact as well and have positive distance. Moreover, the functions

$$d : \mathbb{R}^n \rightarrow [0, 1], \quad x \mapsto d(x) = \frac{\text{dist}(x, A)}{\text{dist}(x, A) + \text{dist}(x, B)}$$

and

$$f : \mathbb{R}^n \rightarrow \mathbb{S}^1, \quad x \mapsto f(x) := \begin{cases} e^{i\pi d(x)}, & x \in \Omega, \\ e^{-i\pi d(x)}, & x \in \mathbb{R}^n \setminus \Omega, \end{cases}$$

are continuous. Let $a \in A$ and $b \in B$. Since Ω and $\mathbb{R}^n \setminus \overline{\Omega}$ are connected, so are their closures $\overline{\Omega}$ and $\overline{\mathbb{R}^n \setminus \Omega} = \mathbb{R}^n \setminus \text{int}(\overline{\Omega}) = \mathbb{R}^n \setminus \Omega$. Hence there are continuous curves γ_1, γ_2 , with $\gamma_i(0) = a$, $\gamma_i(1) = b$ with values in $\overline{\Omega}$ and $\mathbb{R}^n \setminus \Omega$, respectively. Since \mathbb{R}^n is simply connected, there exists a continuous homotopy $H : [0, 1]^2 \rightarrow \mathbb{R}^n$, between γ_1 and γ_2 , preserving the endpoints $H(s, 0) = a$ and $H(s, 1) = b$ for all $s \in [0, 1]$. Then, $f \circ H(s, t)$ is a homotopy of curves in $\mathbb{S}^1 \subset \mathbb{C}$ with fixed end points $f \circ H(i, k) = f(\gamma_i(k)) = (-1)^k$, $i = 1, 2$, $k = 0, 1$ but $f \circ H(0, [0, 1]) = \{z \in \mathbb{S}^1 : \Im(z) \geq 0\}$ and $f \circ H(1, [0, 1]) = \{z \in \mathbb{S}^1 : \Im(z) \leq 0\}$. However, such a homotopy cannot exist. It contradicts, for instance, the Cauchy integral theorem because the corresponding winding numbers

$$\frac{1}{2\pi i} \int_{f \circ H(k, \cdot)} \frac{1}{z} dz = \begin{cases} \frac{1}{2}, & k=0, \\ -\frac{1}{2}, & k=1, \end{cases}$$

are not equal. Hence, $\partial\Omega$ must be connected which completes the proof. \square

The following elementary Lemma allows to partition the domain of $f \in L^1(\mathbb{R})$ into two sets X and X^c such that the values of f are larger on X than on X^c .

Lemma B.4. *Let $f \in L^1(\mathbb{R})$ with $f \geq 0$ and let $M > 0$. Then there exists a measurable set $X \subset \mathbb{R}$ with $|X| = M$ and $\gamma \in \mathbb{R}$ such that $f \geq \gamma$ on X and $f \leq \gamma$ on $\mathbb{R} \setminus X$.*

Proof of Lemma B.4. To simplify the notation we introduce the abbreviation $\{f > t\} := \{x \in \mathbb{R} \mid f(x) > t\}$ and similarly for \geq or $=$ instead of $>$. Consider the monotonically decreasing function $g(t) := |\{f > t\}|$ and define

$$\gamma := \inf\{t > 0 \mid g(t) \leq M\}. \quad (\text{B.11})$$

Chebychef's inequality $g(t) \leq \frac{1}{t} \int_{\mathbb{R}} f(x) dx$ assures that the set on the right hand side of (B.11) is not empty. Using the continuity of the Lebesgue measure, we obtain

$$g(\gamma) = |\{f > \gamma\}| = |\cup_{n \in \mathbb{N}} \{f > \gamma + 1/n\}| = \lim_{n \rightarrow \infty} g(\gamma + 1/n) \stackrel{(B.11)}{\leq} M. \quad (B.12)$$

If $g(\gamma) = M$ the claim follows for $X := \{f > \gamma\}$. However, in the case that $g(\gamma) < M$ we have to add a subset of $\{f = \gamma\}$ to X . To this end, we claim that $|\{f \geq \gamma\}| \geq M$. Indeed, if $\gamma = 0$ this follows directly from $f \geq 0$. For $\gamma > 0$ we get from (B.11) that $g(\gamma - 1/n) > M$ for all $n \in \mathbb{N}$. By Chebycheff's inequality, $g(\gamma - 1/n)$ is finite for $n\gamma > 1$ and we conclude

$$|\{f \geq \gamma\}| = |\cap_{n \in \mathbb{N}} \{f > \gamma - 1/n\}| = \lim_{n \rightarrow \infty} g(\gamma - 1/n) \geq M. \quad (B.13)$$

To define the set X we consider the continuous function

$$h(s) := \lambda_1((-s, s) \cap \{f = \gamma\}).$$

Since $h(0) = 0$ and

$$\begin{aligned} \lim_{s \rightarrow \infty} h(s) &= \lim_{s \rightarrow \infty} |(-s, s) \cap \{f \geq \gamma\}| - \lim_{s \rightarrow \infty} |(-s, s) \cap \{f > \gamma\}| \\ &\stackrel{(B.13)}{\geq} M - g(\gamma), \end{aligned}$$

the mean value theorem implies that there is $s^* \in [0, \infty)$ such that

$$h(s^*) = |(-s^*, s^*) \cap \{f = \gamma\}| = M - g(\gamma). \quad (B.14)$$

The claim now follows for

$$X := \{f > \gamma\} \cup ((-s^*, s^*) \cap \{f = \gamma\}).$$

□

Bibliography

- [1] M. ABRAMOWITZ AND I. A. STEGUN, eds., *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, United States Department of Commerce, National Bureau of Standards (NBS), 10th ed., 1972.
- [2] E. ACERBI, N. FUSCO, AND M. MORINI, *Minimality via second variation for a nonlocal isoperimetric problem*, Communications in Mathematical Physics, 322 (2013), pp. 515–557.
- [3] A. AHARONI, *Energy of one-dimensional domain walls in ferromagnetic films*, Journal of Applied Physics, 37 (1966), pp. 3271–3279.
- [4] G. ANZELLOTTI AND S. BALDO, *Asymptotic development by Γ -convergence*, Applied Mathematics and Optimization, 27 (1993), pp. 105–123.
- [5] G. ANZELLOTTI, S. BALDO, AND A. VISINTIN, *Asymptotic behavior of the Landau-Lifshitz model of ferromagnetism*, Applied Mathematics and Optimization, 23 (1991), pp. 171–192.
- [6] S. D. BADER AND S. S. P. PARKIN, *Spintronics*, Annual Review of Condensed Matter Physics, 1 (2010), pp. 71–88.
- [7] F. BETHUEL, *The approximation problem for Sobolev maps between two manifolds*, Acta Mathematica, 167 (1991), pp. 153–206.
- [8] F. BETHUEL AND X. ZHENG, *Density of smooth functions between two manifolds in Sobolev spaces*, Journal of Functional Analysis, 80 (1988), pp. 60–75.

- [9] A. BOGDANOV AND A. HUBERT, *Thermodynamically stable magnetic vortex states in magnetic crystals*, Journal of Magnetism and Magnetic Materials, 138 (1994), pp. 255–269.
- [10] M. BONACINI AND R. CRISTOFERI, *Local and global minimality results for a nonlocal isoperimetric problem on \mathbb{R}^n* , SIAM Journal on Mathematical Analysis, 46 (2014), pp. 2310–2349.
- [11] M. BONACINI AND H. KNÜPFER, *Ground states of a ternary system including attractive and repulsive Coulomb-type interactions*, Calculus of Variations and Partial Differential Equations, 55 (2016).
- [12] A. BRAIDES AND L. TRUSKINOVSKY, *Asymptotic expansions by Γ -convergence*, Continuum Mechanics and Thermodynamics, 20 (2008), pp. 21–62.
- [13] W. F. BROWN, *Micromagnetics*, vol. 18 of Interscience Tracts of Physics and Astronomy, Interscience Publishers (Wiley & Sons), 1963.
- [14] S. S. CAIRNS, *An elementary proof of the Jordan-Schoenflies theorem*, Proceedings of the American Mathematical Society, 2 (1951), pp. 860–867.
- [15] G. CARBOU, *Thin layers in micromagnetism*, Mathematical Models and Methods in Applied Sciences, 11 (2001), pp. 1529–1546.
- [16] M. CHERMISI AND C. B. MURATOV, *One-dimensional Néel walls under applied external fields*, Nonlinearity, 26 (2013), pp. 2935–2950.
- [17] R. CHOKSI, S. CONTI, R. V. KOHN, AND F. OTTO, *Ground state energy scaling laws during the onset and destruction of the intermediate state in a type I superconductor*, Communications on Pure and Applied Mathematics, 61 (2008), pp. 595–626.
- [18] R. CHOKSI AND R. V. KOHN, *Bounds on the micromagnetic energy of a uniaxial ferromagnet*, Communications on Pure and Applied Mathematics, 51 (1998), pp. 259–289.

- [19] R. CHOKSI, R. V. KOHN, AND F. OTTO, *Domain branching in uniaxial ferromagnets: A scaling law for the minimum energy*, Communications in Mathematical Physics, 201 (1999), pp. 61–79.
- [20] M. CICALESE AND E. SPADARO, *Droplet minimizers of an isoperimetric problem with long-range interactions*, Communications on Pure and Applied Mathematics, 66 (2013), pp. 1298–1333.
- [21] D. COLLIN, G. K. AUERNHAMMER, O. GAVAT, P. MARTINOTY, AND H. R. BRAND, *Frozen-in magnetic order in uniaxial magnetic gels: Preparation and physical properties*, Macromolecular Rapid Communications, 24 (2003), pp. 737–741.
- [22] N. CONDETTE, *Pattern formation in magnetic thin films: Analysis and numerics*, PhD thesis, Humboldt University Berlin, 2010.
- [23] S. CONTI, B. NIETHAMMER, AND F. OTTO, *Coarsening rates in off-critical mixtures*, SIAM Journal on Mathematical Analysis, 37 (2006), pp. 1732–1741.
- [24] E. DE GIORGI, *Nuovi teoremi relativi alle misure $(r - 1)$ -dimensionali in uno spazio ad r dimensioni*, Ricerche di Matematica, 4 (1955), pp. 95–113. English translation in *Ennio De Giorgi, Selected Papers*, L. Ambrosio et al., eds. Springer-Verlag Berlin Heidelberg, 2007.
- [25] E. DE GIORGI, *Frontiere orientate di misura minima*, Seminario di Matematica della Scuola Normale Superiore di Pisa, Editrice tecnico scientifica, 1961.
- [26] A. DESIMONE, H. KNÜPFER, AND F. OTTO, *2-d stability of the Néel wall*, Calculus of Variations and Partial Differential Equations, 27 (2006), pp. 233–253.
- [27] A. DESIMONE, R. V. KOHN, S. MÜLLER, AND F. OTTO, *A reduced theory for thin-film Micromagnetics*, Communications on Pure and Applied Mathematics, 55 (2002), pp. 1408–1460.
- [28] A. DESIMONE, R. V. KOHN, S. MÜLLER, AND F. OTTO, *Recent analytical developments in Micromagnetics*, in The Science of Hysteresis,

- G. Bertotti and I. Mayergoyz, eds., vol. 2 of 3, Academic Press, 2005, ch. 4, pp. 269–381.
- [29] W. F. DRUYVESTEYN AND J. W. F. DORLEIJN, *Calculations of some periodic magnetic domain structures; consequences for bubble devices*, Philips Research Reports, 26 (1971), pp. 11–28.
- [30] L. C. EVANS, *Partial Differential Equations: Second Edition*, Graduate Studies in Mathematics, American Mathematical Society, 2010.
- [31] L. C. EVANS AND R. F. GARIEPY, *Measure Theory and Fine Properties of Functions*, CRC Press, Taylor & Francis Group, 2015.
- [32] I. FERRIER-BARBUT, H. KADAU, M. SCHMITT, M. WENZEL, AND T. PFAU, *Observation of quantum droplets in a strongly dipolar Bose gas*, Physical Review Letters, 116 (2016), p. 215301.
- [33] R. L. FRANK AND E. H. LIEB, *A compactness lemma and its application to the existence of minimizers for the Liquid Drop Model*, SIAM Journal on Mathematical Analysis, 47 (2015), pp. 4436–4450.
- [34] D. B. FUKS AND V. A. ROCHLIN, *Beginner's Course in Topology*, Universitext, Springer-Verlag Berlin Heidelberg, 1984.
- [35] C. J. GARCIA CERVERA, *Magnetic domains and magnetic domain walls*, PhD thesis, New York University, 1999.
- [36] M. GIAQUINTA AND S. HILDEBRANDT, *Calculus of Variations I*, Springer-Verlag Berlin Heidelberg, 2004.
- [37] D. GILBARG AND N. S. TRUDINGER, *Elliptic partial differential equations of second order*, Springer-Verlag Berlin Heidelberg, 2001.
- [38] G. GIOIA AND R. D. JAMES, *Micromagnetics of very thin films*, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science, 453 (1997), pp. 213–223.
- [39] G. GIRAUD, *Sur certains problèmes non linéaires de Neumann et sur certains problèmes non linéaires mixtes*, Annales scientifiques de l'École Normale Supérieure, 49 (1932), pp. 1–104.

- [40] E. GIUSTI, *Minimal Surfaces and Functions of Bounded Variation*, vol. 80 of Monographs in Mathematics, Birkhäuser, 1st ed., 1984.
- [41] F. HANG AND F.-H. LIN, *Topology of Sobolev mappings*, Mathematical Research Letters, 8 (2001), pp. 321–330.
- [42] W. HEISENBERG, *Zur Theorie des Ferromagnetismus*, Zeitschrift für Physik, 49 (1928), pp. 619–636.
- [43] J. HUANG, L. WU, M. CHEN, T. WU, J. WU, Y. HUANG, C. LEE, AND C. FU, *Perpendicular magnetic anisotropy and magnetic domain structure of unpatterned and patterned Co/Pt multilayers*, Journal of Magnetism and Magnetic Materials, 209 (2000), pp. 90–94.
- [44] A. HUBERT AND R. SCHÄFER, *Magnetic domains: The analysis of magnetic microstructures*, Springer-Verlag Berlin Heidelberg, 1998.
- [45] R. IGNAT, *Two-dimensional unit-length vector fields of vanishing divergence*, Journal of Functional Analysis, 262 (2012), pp. 3465–3494.
- [46] R. IGNAT AND H. KNÜPFER, *Vortex energy and 360° Néel walls in thin-film Micromagnetics*, Communications on Pure and Applied Mathematics, 63 (2010), pp. 1677–1724.
- [47] R. IGNAT AND F. OTTO, *A compactness result in thin-film Micromagnetics and the optimality of the Néel wall*, Journal of the European Mathematical Society, 10 (2008), pp. 909–956.
- [48] S. I. IWASAKI, *Perpendicular magnetic recording—its development and realization*, Journal of Magnetism and Magnetic Materials, 324 (2012), pp. 244–247.
- [49] W. JIANG, P. UPADHYAYA, W. ZHANG, G. YU, M. B. JUNGFLEISCH, F. Y. FRADIN, J. E. PEARSON, Y. TSERKOVNYAK, K. L. WANG, O. HEINONEN, S. G. E. TE VELTHUIS, AND A. HOFFMANN, *Blowing magnetic Skyrmion bubbles*, Science, 349 (2015), pp. 283–286.
- [50] V. JULIN, *Remark on a nonlocal isoperimetric problem*, Nonlinear Analysis: Theory, Methods & Applications, 154 (2017), pp. 174–188.

- [51] B. KAPLAN AND G. GEHRING, *The domain structure in ultrathin magnetic films*, Journal of Magnetism and Magnetic Materials, 128 (1993), pp. 111 – 116.
- [52] C. KITTEL, *Theory of the structure of ferromagnetic domains in films and small particles*, Physical Review, 70 (1946), pp. 965–971.
- [53] C. KITTEL, *Physical theory of ferromagnetic domains*, Reviews of Modern Physics, 21 (1949), pp. 541–583.
- [54] H. KNÜPFER AND R. V. KOHN, *Minimal energy for elastic inclusions*, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science, 467 (2011), pp. 695–717.
- [55] H. KNÜPFER, R. V. KOHN, AND F. OTTO, *Nucleation barriers for the cubic-to-tetragonal phase transformation*, Communications on Pure and Applied Mathematics, 66 (2013), pp. 867–904.
- [56] H. KNÜPFER AND C. B. MURATOV, *Domain structure of bulk ferromagnetic crystals in applied fields near saturation*, Journal of Nonlinear Science, 21 (2011), pp. 921–962.
- [57] H. KNÜPFER AND C. B. MURATOV, *On an isoperimetric problem with a competing nonlocal term I: The planar case*, Communications on Pure and Applied Mathematics, 66 (2013), pp. 1129–1162.
- [58] H. KNÜPFER AND C. B. MURATOV, *On an isoperimetric problem with a competing nonlocal term II: The general case*, Communications on Pure and Applied Mathematics, 67 (2013), pp. 1974–1994.
- [59] R. V. KOHN, *Energy-driven pattern formation*, in Proceedings of the International Congress of Mathematicians, Madrid 2006, vol. 1, European Mathematical Society Publishing House, 2007, pp. 359–383.
- [60] R. V. KOHN AND V. V. SLASTIKOV, *Another thin-film limit of Micromagnetics*, Archive for Rational Mechanics and Analysis, 178 (2005), pp. 227–245.

- [61] C. KOOY AND U. ENZ, *Experimental and theoretical study of the domain configuration in thin layers of $\text{BaFe}_{12}\text{O}_{19}$* , Philips Research Reports, 15 (1960), pp. 7–29.
- [62] M. KRONSEDER, T. MEIER, M. ZIMMERMANN, M. BUCHNER, M. VOGEL, AND C. BACK, *Real-time observation of domain fluctuations in a two-dimensional magnetic model system*, Nature Communications, 6 (2015).
- [63] M. KURZKE, *Boundary vortices in thin magnetic films*, Calculus of Variations and Partial Differential Equations, 26 (2006), pp. 1–28.
- [64] L. D. LANDAU AND E. M. LIFSHITZ, *On the theory of the dispersion of magnetic permeability in ferromagnetic bodies*, Physikalische Zeitschrift der Sowjetunion, 8 (1935), pp. 101–114.
- [65] P. L. LIONS, *The concentration-compactness principle in the calculus of variations. The locally compact case, part 1*, Annales de l’I.H.P. Analyse non linéaire, 1 (1984), pp. 109–145.
- [66] J. LU AND F. OTTO, *Nonexistence of a minimizer for Thomas-Fermi-Dirac-von Weizsäcker model*, Communications on Pure and Applied Mathematics, 67 (2013), pp. 1605–1617.
- [67] F. MAGGI, *Sets of finite perimeter and geometric variational problems: An introduction to Geometric Measure Theory*, Cambridge University Press, 2012.
- [68] J. C. MAXWELL, *A Treatise on Electricity and Magnetism*, vol. 2, Dover Publications, 1954. (unabridged, slightly altered, republication of the 3rd edition, originally published by Clarendon, Oxford, 1891).
- [69] J. MCCORD, Website of the Nanoscale Magnetic Materials Group, Institute for Materials Science, Kiel University. URL: <http://www.tf.uni-kiel.de/matwis/nmm/en/gallery/ferrite-garnet-film>, (accessed 10.7.2017).
- [70] C. MELCHER, *Chiral skyrmions in the plane*, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science, 470 (2014), p. 20140394.

- [71] S. G. MIKHLIN, *Mathematical physics: An advanced course*, North-Holland Publishing, 1970.
- [72] C. MIRANDA, *Sulle proprietà di regolarità di certe trasformazioni integrali*, in *Memorie della Classe di Scienze Fisiche, Matematiche e Naturali*, vol. 7, ser. 8, Accademia Nazionale dei Lincei, 1965, pp. 303–363.
- [73] C. MIRANDA, *Partial differential equations of elliptic type*, Springer-Verlag Berlin Heidelberg, 2012.
- [74] R. MOSER, *Boundary vortices for thin ferromagnetic films*, *Archive for Rational Mechanics and Analysis*, 174 (2004), pp. 267–300.
- [75] C. B. MURATOV, *A universal thin film model for Ginzburg-Landau energy with dipolar interaction*, arXiv preprint arXiv:1702.01986, (2017).
- [76] C. B. MURATOV AND V. V. SLASTIKOV, *Domain structure of ultrathin ferromagnetic elements in the presence of Dzyaloshinskii-Moriya interaction*, *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science*, 473 (2016), p. 20160666.
- [77] D. NAVAS, C. REDONDO, G. A. BADINI CONFALONIERI, F. BATALLAN, A. DEVISHVILI, O. IGLESIAS-FREIRE, A. ASENJO, C. A. ROSS, AND B. P. TOPERVERG, *Domain-wall structure in thin films with perpendicular anisotropy: Magnetic force microscopy and polarized neutron reflectometry study*, *Physical Review B*, 90 (2014), p. 54425.
- [78] K.-O. NG AND D. VANDERBILT, *Stability of periodic domain structures in a two-dimensional dipolar model*, *Physical Review B*, 52 (1995), pp. 2177–2183.
- [79] L. NÉEL, *Les lois de l'aimantation et de la subdivision en domaines élémentaires d'un monocristal de fer*, *Journal de Physique et Le Radium*, 5 (1944), pp. 241–251.
- [80] L. NÉEL, *Les lois de l'aimantation et de la subdivision en domaines élémentaires d'un monocristal de fer*, *Journal de Physique et Le Radium*, 5 (1944), pp. 265–276.

- [81] F. OTTO AND T. VIEHMANN, *Domain branching in uniaxial ferromagnets: Asymptotic behavior of the energy*, Calculus of Variations and Partial Differential Equations, 38 (2009), pp. 135–181.
- [82] S. RIGOT, *Ensembles quasi-minimaux avec contrainte de volume et rectifiabilité uniforme*, Mémoires de la Société Mathématique de France, 82 (2000), pp. 1–104.
- [83] S. ROHART AND A. THIAVILLE, *Skyrmion confinement in ultrathin film nanostructures in the presence of Dzyaloshinskii-Moriya interaction*, Physical Review B, 88 (2013), p. 184422.
- [84] N. SARATZ, U. RAMSPERGER, A. VINDIGNI, AND D. PESCIA, *Irreversibility, reversibility, and thermal equilibrium in domain patterns of Fe films with perpendicular magnetization*, Physical Review B, 82 (2010), p. 184416.
- [85] M. SCHMITT, M. WENZEL, F. BÖTTCHER, I. FERRIER-BARBUT, AND T. PFAU, *Self-bound droplets of a dilute magnetic quantum liquid*, Nature, 539 (2016), pp. 259–262.
- [86] M. SCHOTT, A. BERNAND-MANTEL, L. RANNO, S. PIZZINI, J. VOGEL, H. BÉA, C. BARADUC, S. AUFFRET, G. GAUDIN, AND D. GIVORD, *The skyrmion switch: Turning magnetic skyrmion bubbles on and off with an electric field*, Nano Letters, 17 (2017), pp. 3006–3012.
- [87] A. SOUMYANARAYANAN, M. RAJU, A. L. GONZALEZ OYARCE, A. K. C. TAN, M.-Y. IM, A. P. PETROVIC, P. HO, K. H. KHOO, M. TRAN, C. K. GAN, F. ERNULT, AND C. PANAGOPOULOS, *Tunable room temperature magnetic skyrmions in Ir/Fe/Co/Pt multilayers*, arXiv preprint arXiv:1606.06034, (2016).
- [88] M. SPECKMANN, H. P. OEPEN, AND H. IBACH, *Magnetic domain structures in ultrathin Co/Au(111): On the influence of film morphology*, Physical Review Letters, 75 (1995), pp. 2035–2038.
- [89] E. M. STEIN, *Singular integrals and differentiability properties of functions*, Princeton University Press, 1970.

- [90] E. C. STONER, *XCVII. The demagnetizing factors for ellipsoids*, The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, 36 (1945), pp. 803–821.
- [91] I. TAMANINI, *Boundaries of Caccioppoli sets with Hölder-continuous normal vector*, Journal für die reine und angewandte Mathematik, 334 (1982), pp. 27–39.
- [92] A. THIAVILLE, D. TOMÁŠ, AND J. MILTAT, *On corner singularities in Micromagnetics*, Physica Status Solidi (a), 170 (1998), pp. 125–135.
- [93] G. VERCHOTA, *Layer potentials and regularity for the Dirichlet problem for Laplace’s equation in Lipschitz domains*, Journal of Functional Analysis, 59 (1984), pp. 572 – 611.
- [94] S. WOO, K. LITZIUS, B. KRUGER, M.-Y. IM, L. CARETTA, K. RICHTER, M. MANN, A. KRONE, R. M. REEVE, M. WEIGAND, P. AGRAWAL, I. LEMESH, M.-A. MAWASS, P. FISCHER, M. KLAUI, AND G. S. D. BEACH, *Observation of room-temperature magnetic skyrmions and their current-driven dynamics in ultrathin metallic ferromagnets*, Nature Materials, 15 (2016), pp. 501–506.
- [95] M. YAMANOUCHI, A. JANDER, P. DHAGAT, S. IKEDA, F. MATSUKURA, AND H. OHNO, *Domain structure in CoFeB thin films with perpendicular magnetic anisotropy*, IEEE Magnetism Letters, 2 (2011), p. 3000304.
- [96] D. ZWILLINGER, ed., *CRC Standard Mathematical Tables and Formulae*, CRC Press, Taylor & Francis Group, 32nd ed., 2011.